2.4 Sequences and Summations

A ~~sequence~~ is a function from a subset S of $\mathbb{Z}$. The domain is ussually $\mathbb{N} = \{0,1,2,\ldots\}$ or $\mathbb{Z}^+ = \{1,2,\ldots\}$.

$S \subseteq \mathbb{Z}$ write a_n for $n \in S$

$\{a_n\}$ (sequence, NOT a set)

$\{a_n\}_{n \in S}$

Examples

1) $\{\frac{1}{n}\}_{n \in \mathbb{Z}^+}$ $\quad \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

$a_n = \frac{1}{n} \quad a_1 = 1, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3}$

2) $\{n^2\}_{n \in \mathbb{N}}$ $\quad 0, 1, 4, 9, 16, \ldots$

$a_n = n^2 \quad a_5 = 25, \ a_9 = 81$

A sequence $a_0, ar, ar^2, ar^3, \ldots$ is called a geometric progression

A sequence $a, a+d, a+2d, a+3d, \ldots$ is called an arithmentic progression

Examples

$a_n = 3 \cdot 2^{n+1} \quad n \in \mathbb{N}$

$a_0 = 6, \; a_1 = 12, \; a_2 = 24, \ldots$

$b_n = 6n + 1 \quad n \in \mathbb{N}$

$b_0 = 1, \; b_1 = 7, \; b_2 = 13, \ldots$

Arithmetic, geometric, or neither?

$12, 4, 4/3, 4/9, \ldots$

$2, -4, 8, -16, 32, \ldots$

$5, 7, 10, 14, 19, \ldots$

$3, 8, 13, 18, 23, 28, \ldots$

A <u>recurrence relation</u> of a sequence $\{a_n\}$ is an equation expressing a_n in terms of previous elements of the sequence.

$\{a_n\}$ $\quad a_n = a_{n-1} + 5$, $a_0 = 3$
$n > 0$

3, 8, 13, 18, ...

$\{b_n\}$ $\quad b_n = -2b_{n-1}$ $\quad b_0 = 2$
$n > 0$

2, -4, 8, -16, ...

$\{F_n\}$ $\quad F_1 = 1, F_2 = 1$ $\quad F_n = F_{n-1} + F_{n-2}$ $n \geq 3$

1, 1, 2, 3, 5, 8, 13, ... $\quad$ Fibonacci numbers

A <u>closed formula</u> is an explicit formula for terms. A closed formula may be called a "solution" ~~for~~ a recurrence relation.

$a_1 = 4 \quad a_n = 2a_{n-1} - a_{n-2} \quad n \geq 2$
$a_0 = 0$

Show $a_n = 4n$ is a solution

$a_0 = 4 \cdot 0$ ✓ $\quad a_n = 2a_{n-1} - a_{n-2}$
$a_1 = 4 \cdot 1$ ✓ $\quad = 2\,4(n-1) - 4(n-2)$
$= 8n - 8 - 4n + 8$
$= 4n$

If $a_{n-1} = 4(n-1)$ and $a_{n-2} = 4(n-2)$, then $a_n = 4n$

Factorials obey a recurrence relation

$n! = n\cdot(n-1)(n-2)\cdots 2\cdot 1 \qquad n \geq 1$

$0! = 0$

$3! = 3\cdot 2\cdot 1 = 6$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$n! = n\cdot(n-1)!$.

If $a_n = n!$, then $a_n = n a_{n-1}$ w/ $a_0 = 1$.
$n \geq 1$

we can sum the terms of a sequence

$\{a_n\}$

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

$\{a_n\} \quad a_n = n^2 \quad n \in \mathbb{N}$

$$\sum_{i=0}^{4} a_i = 0 + 1 + 4 + 9 + 16 = 30$$

$$\sum_{i=9}^{10} a_i = 81 + 100 = 181$$

$$\sum_{i=7}^{13} a_i = \sum_{j=7}^{13} a_j$$

$\{ar^n\}$ geometric progression

$$\sum_{j=0}^{n} ar^j = \begin{cases} \frac{ar^{n+1}-a}{r-1} & r \neq 1 \\ (n+1)\,a & r = 1 \end{cases}$$

$$\sum_{j=0}^{3} 2^j = 1+2+4+8 = 15 = \frac{16-1}{2-1}$$

Other summation formulas

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

$$\sum_{k=1}^{10} k(k+2) = \sum_{k=1}^{10} (k^2+2k) = \sum_{k=1}^{10} k^2 + 2\sum_{k=1}^{10} k$$

$$= \frac{(10)(11)(21)}{6} + 2\left(\frac{(10)(11)}{2}\right)$$

We can have double/nested summation

$$\sum_{i=1}^{3} \sum_{j=2}^{5} i^2 (j+1)$$

$$= (3+4+5+6) + (4(3)+4(4)+4(5)+4(6)) + (9(3)+9(4)+9(5)+9(6))$$

Can think in terms of loops

for $i=1$ to $i=3$
  for $j=2$ to $j=5$
    ⋮