2.5 Cardinality of Sets

Two sets have the same cardinality if and if only there is a bijection between them.

$|A| = |B|$ bijection $f: A \rightarrow B$
bijection $f^{-1}: B \rightarrow A$

$|A| \leq |B|$ injection $f: A \rightarrow B$

A set S is countablely infinite if $|S| = |\mathbb{Z}^+|$.

$S = \{2n : n \in \mathbb{Z}\}$ is countable

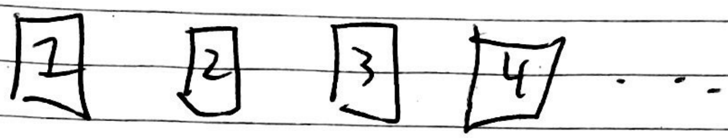
0	2	-2	4	-4	8	-8	...
$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
1	2	3	4	5	6	7	

$f: \mathbb{Z}^+ \rightarrow S$

$f(n) = -n+1$ n odd

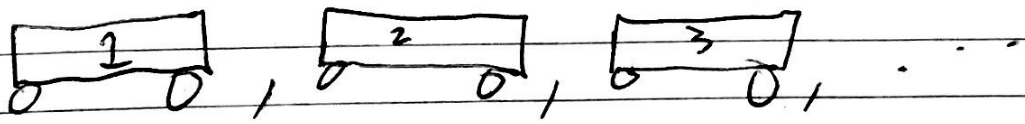
$f(n) = n$ n even

Hilbert's Grand Hotel is a hotel with a countably infinity number of room



Every room of Hilbert's Grand Hotel is occupied. How can we fit a new guest without evicting any current guest?

~~Every room of Hilbert's Grand Hotel is occupied.~~ How can we fit countably infinitely many busses each carrying countably infinitely many guest into Hilbert's Grand Hotel?



$\mathbb{Q}$ is countable

Lets show the set of positive rational numbers is countable.

Make a "box" of all expressions $\frac{a}{b}$ for $a, b \in \mathbb{Z}^+$ so that numerator of first column is 1, second column is 2 etc. And denominator of first row is 1, ...

$$\begin{array}{ccccccc}
 & & 1 & 3 & 5 & 9 & 11 \\
 & & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} \; \cdot \\
1^{st} & 2 & \frac{1}{2} & \frac{2}{2} & \frac{3}{2}\,{}^{8} & \frac{4}{2} & \frac{5}{2} \cdots \\
2^{nd} & 4 & \frac{1}{3} & \frac{2}{3}\,{}^{7} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} \cdots \\
3^{rd} & 6 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} \cdots \\
4^{th} & 10 & \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} \\
5^{th} & & & & & &
\end{array}$$

$\boxed{\text{Thm}}$ The union of two countable sets is countable.

Pf) $A = \{a_1, a_2, \ldots$

$B = \{b_1, b_2, \ldots$

$A \cup B = \{a_1, b_1, a_2, b_2, \ldots\} = \{c_1, c_2, c_3, \ldots$

$c_{2k-1} = a_k \quad c_{2k} = b_k$

Not all sets with infinitely many elements have the same cardinality.

$\mathbb{R}$ is uncountable

$\boxed{\text{Schröder-Bernstein Theorem}}$

If there exists a 1-1 function $f: A \to B$ and a 1-1 function $g: B \to A$, then $|A| = |B|$

Show $|(0,1)| = |\mathbb{R}|$

$f: (0,1) \to \mathbb{R}$ by $f(x) = x$ is 1-1

So we need only find a 1-1 function $g: \mathbb{R} \to (0,1)$

Consider $g_1: \mathbb{R} \to (0, \infty)$ $g_1(x) = e^x$

$g_2: (0,\infty) \to (2,\infty)$ $g_2(x) = e^x + 2$

$g_3: (2,\infty) \mapsto (0,1)$ $g_3(x) = \frac{1}{x}$

$g = g_3 \circ g_2 \circ g_1$

Each of g_1, g_2, g_3 is 1-1 so the composition is 1-1.

It follows $|(0,1)| = |\mathbb{R}|$ by the Schröder-Bernstein Theorem.