3.2 The Growth of Functions

$f, g: \mathbb{Z} \to \mathbb{R}$
$f, g: \mathbb{N} \to \mathbb{R}$ functions
$f, g: \mathbb{Z}^+ \to \mathbb{R}$

We say $f(x)$ is $O(g(x))$ ("big-O of") if there exists constants C and k such that

$$|f(x)| \leq C|g(x)| \quad \forall x > k$$

We think of this as the function $f(x)$ growing no faster than $g(x)$.

Example showing big-O

Show x^2+1 is $O(x^2)$

Need to find the constants C and k from definition.

$0 \leq x^2+1 \leq x^2+x^2 \leq 2x^2$ for any $x > 0$

so $|x^2+1| \leq 2|x^2|$ for all $x > 0$ and x^2+1 is $O(x^2)$ with $C=2$ and $k=1$

Example showing NOT big-O

Show x^3+1 is NOT $O(x^2)$

Assume x^3+1 is $O(x^2)$ so $|x^3+1| \leq C|x^2|$ $\forall x > k$.
Choose $x > C$, then

$$x^3+1 > x^3 > C \cdot x^2$$

which is a contradiction

$\boxed{\text{Thm}}$ If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x_1 + a_0$, then $f(x)$ is $O(x^n)$

Pf) $|f(x)| = |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x_1 + a_0|$

when $x > 1$ $\leq |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0|$

$\leq |a_n| x^n + |a_{n-1}| x^n + \cdots + |a_1| x^n + |a_0| x^n$

$= (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|) x^n$

So $O(x^n)$ with $C = |a_n| + \cdots + |a_0|$ and $k = 1$.

Show $f(n) = n!$ is $O(n^n)$.

$n! = n(n-1)(n-2)\cdots(2)(1)$

$\leq n(n)(n)\cdots(n)(n)$

$= n^n$ for $n \geq 1$

So $n!$ is $O(n^n)$ since

$|n!| \leq 1 |n^n|$ $\forall$ $n \geq 1$

$C = 1, k = 1$

General order of functions

$1, \log(n), n, n\log(n), n^2, n^3, \ldots, 2^n, 3^n, \ldots, n!, n^n$

Show 2^n is $O(3^n)$ but 3^n is NOT $O(2^n)$

Since $2 < 3$ we have $2^n < 3^n \; \forall n \geq 1$
so 2^n is $O(3^n)$

Assume 3^n was $O(2^n)$ then there would be constants C and K so that

$$3^n \leq C \cdot 2^n \quad \text{for } n > K$$

This would mean

$$\left(\frac{3}{2}\right)^n < C \quad \text{for } n > K$$

but this is a contradiction since

$$\left(\frac{3}{2}\right)^n \geq C \quad \text{for } n \geq \log_{\frac{3}{2}}(C)$$

We can talk about big-O for combinations of functions

$\boxed{\text{THM}}$ If $f_1(x)$ and $f_2(x)$ are $O(g(x))$, then $(f_1+f_2)(x)$ is $O(g(x))$

Pf) $|f_1(x)| \leq C_1 |g(x)| \quad \forall\, x \geq k_1$

$|f_2(x)| \leq C_2 |g(x)| \quad \forall\, x \geq k_2$

use triangle inequality

$\Rightarrow |(f_1+f_2)(x)| \leq (C_1 + C_2)|g(x)| \quad \forall\, x \geq \max(k_1, k_2)$

So $(f_1+f_2)(x)$ is $O(g(x))$ with $C = C_1 + C_2$ and $k = \max(k_1, k_2)$

$\boxed{\text{THM}}$ If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 f_2)(x)$ is $O((g_1 g_2)(x))$

Pf) $|f_1(x)| \leq C_1 |g_1(x)| \quad \forall\, x \geq k_1$

$|f_2(x)| \leq C_2 |g_2(x)| \quad \forall\, x \geq k_2$

$\Rightarrow |f_1 f_2(x)| = |f_1(x)||f_2(x)| \leq C_1 C_2 |g_1(x) g_2(x)| \quad \forall\, x \geq k$

So $(f_1 f_2)(x)$ is $O((g_1 g_2)(x))$ with

$C = C_1 C_2$ and $k = \max(k_1, k_2)$

$f(x)$ is $\Omega(g(x))$ ("big-Omega"), if there are constants C and k with $C>0$ and $|f(x)| \geq C|g(x)|$ whenever $x>k$

$f(x)$ is $\Theta(g(x))$ ("big-Theta"), if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$.

Example

Let $f(n) = \sum_{j=1}^{n} j = 1+2+\cdots+n$

$f(n) = \frac{n(n+1)}{2} = \frac{1}{2}(n^2+n)$

$f(n)$ is $\Theta(n^2)$ let's see why

$f(n)$ is $O(n^2)$ by a previous theorem

$f(n)$ is $\Omega(n^2)$ because

$$1+2+\cdots+n = \left(1+2\cdots\left\lceil \tfrac{n}{2}\right\rceil-1\right) + \left(\left\lceil \tfrac{n}{2}\right\rceil + \left(\left\lceil \tfrac{n}{2}\right\rceil+1\right) + \cdots + n\right)$$

$$\geq \left\lceil \tfrac{n}{2}\right\rceil + \left(\left\lceil \tfrac{n}{2}\right\rceil+1\right) + \cdots + n$$

$$\geq \left\lceil \tfrac{n}{2}\right\rceil + \left\lceil \tfrac{n}{2}\right\rceil + \cdots + \left\lceil \tfrac{n}{2}\right\rceil$$

$$= \left(n - \left(\left\lceil \tfrac{n}{2}\right\rceil - 1\right)\right)\left\lceil \tfrac{n}{2}\right\rceil$$

$$= \left(n - \left\lceil \tfrac{n}{2}\right\rceil + 1\right)\left\lceil \tfrac{n}{2}\right\rceil$$

$$\geq \left(\tfrac{n}{2}\right)^2 = \tfrac{1}{4}n^2$$