# 5.1 Mathematical Induction

Want to prove infinitely many statements $P(n)$ for $n = n_0, n_0+1, n_0+2, \ldots$

Plan

1) Verify $P(n_0)$ is true

2) Show the ~~conditional~~ conditional statement
$$P(k) \longrightarrow P(k+1)$$
is true

A first example is to prove a summation formula from section 2.4.

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \quad \text{for any } n \in \mathbb{Z}^+$$

Here $P(n)$ is the statement $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ and $n_0 = 1$.

1) $P(1)$ says $\sum_{j=1}^{1} j = 1 = \frac{1(2)}{2}$ which is true

2) Assume $P(k)$ is true for some $k \geq 1$ so
$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}$$

Then
$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1) = \frac{k(k+1)}{2} + (k+1)$$
$$= \frac{k(k+1) + 2(k+1)}{2}$$
$$= \frac{(k+2)(k+1)}{2}$$

This show $P(k+1)$ is true after assuming $P(k)$ is true. Hence the conditional statement

$$P(k) \longrightarrow P(k+1)$$

is true.

Therefor $P(n)$ which says $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$ is true for all $n$ by induction.

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Prove $\sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \quad \forall n \geq 0$

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For $n=0$, $\sum_{j=0}^{0} 2^j = 1 = 2^{0+1} - 1$.

Now assume $\sum_{j=0}^{k} 2^j = 2^{k+1} - 1$

$$\sum_{j=0}^{k+1} 2^j = \sum_{j=0}^{k} 2^j + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^{k+1}$$

$$= (2^{k+1} + 2^{k+1}) - 1$$

$$= 2(2^{k+1}) - 1$$

$$= 2^{k+2} - 1$$

$\boxed{\text{Prove } n^2 < 2^n \quad \forall\, n > 4}$

For $n = 5$ $\quad n^2 = 25 < 32 = 2^n$.

Assume $k^2 < 2^k$ for some $k > 4$

$$
\begin{aligned}
(k+1)^2 &= k^2 + 2k + 1 \\
&< 2^k + 2k + 1 \\
&< 2^k + 2^k \\
&= 2^{k+1}
\end{aligned}
$$

$\boxed{\text{Prove } 2n+1 < 2^n \quad \forall\, n \geq 4}$

$n = 5 \quad 11 < 32$

Assume $2k+1 < 2^k$ for some $k$

$2(k+1)$ [illegible] $= 2$ [illegible] $k$

$$2(k+1) + 1 = 2k+1 + 2 < 2^k + 2 < 2^k + 2^k = 2^{k+1}$$

Strong Induction works similar

Want to show $P(1), P(2), P(3), \ldots$

(1) Base case: Show $P(1)$ is true

(2) Inductive step: Show that the conditional

$$[P(1) \wedge P(2) \wedge \cdots \wedge P(k)] \longrightarrow P(k+1)$$

is true

Instead of assume previous one case is true $(P(k) \longrightarrow P(k+1))$ we assume all previous cases are true.

Show if one has 4-cent and 5-cent stamps any amount of postage 12 cents or more can be made.

12 cents can be made with 3 4-cent stamps. 13 cents w/ 4+4+5, 14 cents w/ 4+5+5, 15 cents w/ 5+5+5. Assume all values between 12 cents and k cents can be made. where $k \geq 15$.

Consider $k+1$ cents. Since $k \geq 15$ we have $k-3 \geq 12$. So take are solution for $k-3$ cents and add a 4-cent stamp.

# Nim

2 players
$n$ matches
Each turn players remove 1, 2, or 3 matches
who ever removes last match wins.

Prove the first player can win if $n = 4j, 4j+2, 4j+3$
and second player wins if $n = 4j+1$
($j$ nonnegative integer)

$n=1$ second player wins
$n=2$ first player wins (remove 1)
$n=3$ first player wins (remove 2)
$n=4$ first player wins (remove 3)

Assume result is true for all number of matches smaller than $n$. Consider $n$ matches.

If $n=4j$, remove 3 matches left with $4(j-1)+1$.

If $n=4j+2$ remove 1 match, left with $4j+1$

If $n=4j+3$ remove 2 matches, left with $4j+1$

If $n=4j+1$

| # removed | # left |
|---|---|
| 1 | $4j$ |
| 2 | $4(j-1)+3$ |
| 3 | $4(j-1)+2$ |