Test 2 Review

2.3 #14. Determine if $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto for

(d) $f(m,n) = |m| - |n|$

(e) $f(m,n) = m^2 - 4$

We claim $f(m,n) = |m| - |n|$ is onto. Take $a \in \mathbb{Z}$ arbitrary. If $a \geq 0$ consider $(a,0) \in \mathbb{Z} \times \mathbb{Z}$, then $f(a,0) = a$. If $a < 0$ consider $(0,a) \in \mathbb{Z} \times \mathbb{Z}$, then $f(0,a) = a$. Therefor $f(m,n) = |m| - |n|$ is onto

We clam $f(m,n) = m^2 - 4$ is not onto. To see this take $-5 \in \mathbb{Z}$. Since $m^2 \geq 0$ for any $m \in \mathbb{Z}$ we see that $f(m,n) = m^2 - 4 \geq -4$ for any $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. Therefor $f(m,n) \neq -5$ for any $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ and $f(m,n) = m^2 - 4$ is not onto.

2.4 #1$\cancel{7}$ f) ← ~~Similar but different~~

Find the solution to $a_n = 3a_{n-1} - 1$ for $n \geq 0$ and $a_0 = 1$

$a_0 = 1$

$a_1 = 3 - 1 = 2$

$a_2 = 6 - 1 = 5$

$a_3 = 15 - 1 = 14$

$a_4 = 42 - 1 = 41$

$a_5 = 123 - 1 = 122$

$1 + 1 = 2$

$2 + 3 = 5$

$5 + 9 = 14$

$14 + 27 = 41$

$a_n = a_{n-1} + 3^{n-1}$

$a_n = 3a_{n-1} - 1$

$a_0 = 1$

$a_1 = 1 + 1$

$a_2 = 1 + 1 + 3$

$a_3 = 1 + 1 + 3 + 9$

$a_n = 1 + \sum_{j=0}^{n-1} 3^j$

$$a_n = 1 + \sum_{j=0}^{n-1} 3^j$$

$$a_{n+1} = 3a_n + 1 = 3\left(1 + \sum_{j=0}^{n-1} 3^j\right) - 1$$

$$= 3 + 3\sum_{j=0}^{n-1} 3^j - 1$$

$$= 3 + \sum_{j=0}^{n-1} 3^{j+1} - 1$$

$$= 3 + \sum_{j=1}^{n} 3^j - 1$$

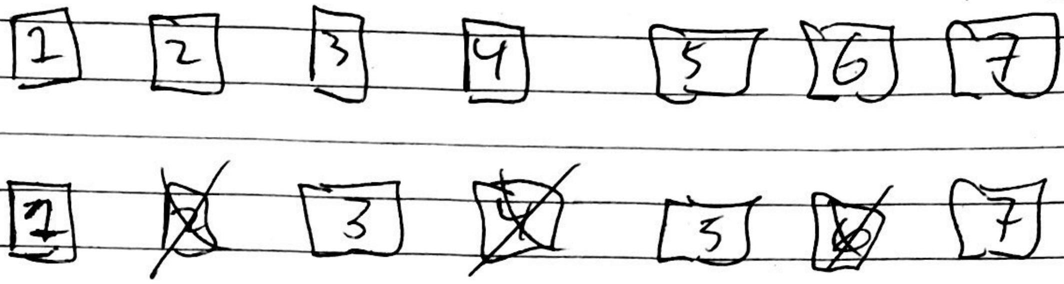
$$= 1 + 1 + \sum_{j=1}^{n} 3^j$$

$$= 1 + \sum_{j=0}^{n} 3^j$$

$$a_n = 1 + \sum_{j=0}^{n-1} 3^j = 1 + \frac{3^n - 1}{2}$$

$a_0 = 1$
$a_1 = 2$
$a_2 = 5$
$a_4 = 14$

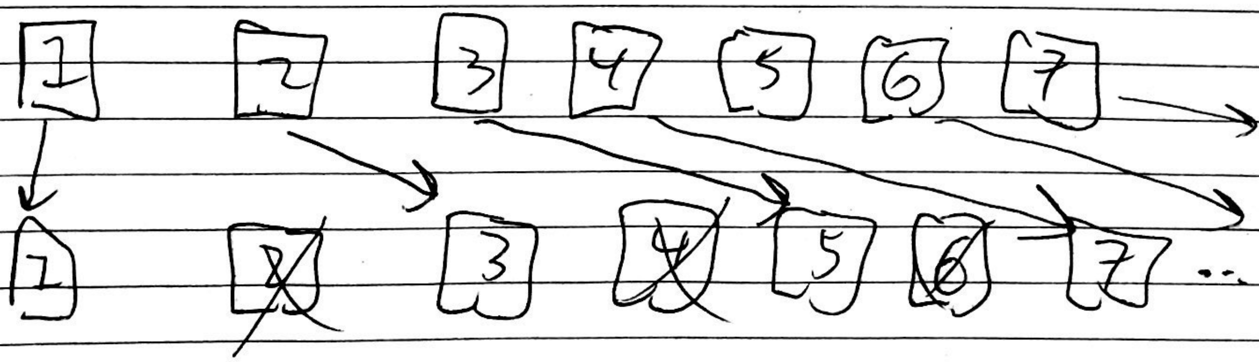
$\boxed{2.5 \ \#14}$ Suppose that Hilberts Grand Hotel is fully occupied but all even numbered rooms are closed for maintence. Show that all guests can remain in the hotel



Need an injective function

$$f: \mathbb{Z}^+ \longrightarrow \{2k-1 : k \in \mathbb{Z}^+\}$$

Let $f(n) = 2n-1$ for each $n \in \mathbb{Z}^+$. That is send ~~each~~ the guest in room n to room $2n-1$.



$\boxed{\#3.2\ \#22}$ Show that $3x^4+1$ is $O(x^4/2)$ and that $x^4/2$ is $O(3x^4+1)$

Follow from a theorem, but we want to do it by definition for fun / practice.

$$3x^4+1 \leq 3x^4+x^4 = 4x^4 = 8(x^4/2)$$

Whenever $x \geq 1$. Also for $x \geq 1$ all expressions are nonnegative so we can add absolute value

$$|3x^4+1| \leq 8|x^4/2| \quad \forall\ x \geq 1$$

So $3x^4+1$ is $O(x^4/2)$

Now for the other way

$$x^4/2 \leq 3x^4 < 3x^4+1 \quad \text{for any } x > 0$$

Again all expressions are nonnegative so

$$|x^4/2| \leq 1 \cdot |3x^4+1| \quad \forall\ x > 0$$

and $x^4/2$ is $O(3x^4+1)$

5-1 #10

Find a formula for $\sum_{k=1}^{n} \frac{1}{k(k+1)}$ and prove it

$n \geq 1$

$\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$, $\sum_{k=1}^{2} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

$\sum_{k=1}^{3} \frac{1}{k(k+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4}$

conjecture: $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

Proof by induction

Base case: $\sum_{k=1}^{1} \frac{1}{k(k+1)} = \frac{1}{2}$.

Inductive step: Assume $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$, then

$$\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k(k+1)} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)}$$

$$= \frac{(n+1)^2}{(n+1)(n+2)}$$

$$= \frac{n+1}{n+2}$$

~~5~~5.2 # 6

Show using 3-cent and 10-cent stamp any amount of 18 cents or greater can be made

$18 = 6 \cdot 3$
$19 = 2 \cdot 3 + 10$
$20 = 2 \cdot 10$

Base cases.

So we can make 18 cents, 19 cents, and 20 cents. Now assume we can make k cents for all $18 \leq k \leq n$ for some $n > 20$.

Consider $n+1$ cents. Since $n > 20$ we know that $(n+1)-3 = n-2 > 18$ also $n-2 < n$. So be strong induct we can make $n-2$ cent. Adding a 3 cent stamp we have now made n cents. The result follows by strong induction.