2.3 Functions

Given two nonempty sets A and B a function f from A to B is an assignment of exactly one element of B to each element of A.

<u>Notation</u>: $f: A \longrightarrow B$ $\quad f(a)=b \quad a\in A,\ b\in B$

A is called the <u>domain</u>.
B is called the <u>codomain</u>.

If $f(a)=b$, then b is called the <u>image</u> of a and a is called a <u>preimage</u> of b.

The <u>range</u> of f is the set of all images

$$\{f(a) : a \in A\}$$

TRUE or FALSE
Let $f: A \to B$ be a function and $a\in A$, $b\in B$

The element $a\in A$ can have more than one image

The element $b\in B$ can have more than one preimage.

Functions can be thought of as <u>relations</u>.
A <u>relation</u> between sets A and B is just a subset $S \subseteq A\times B$.

$f: A \to B$

$$\{(a, f(a)) : a\in A\} \subseteq A\times B$$

Consider the function which maps people to there preferred drink.

$A = \{Alice, Bob, Chuck\}$

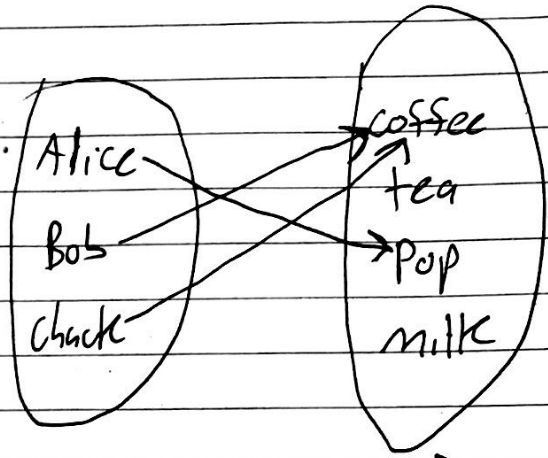
$B = \{coffee, tea, pop, milk\}$

$f: A \to B$

$f(Alice) = pop$

$f(Bob) = coffee$

$f(Chuck) = coffe$



$\{(Alice, pop), (Bob, coffee), (Chuck, coffee)\}$

A programming aside ...

Consider the example function (method) is a Java type languange

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boolean some_function(int a) { ... }
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What is the domain and codomain of this function?

$f: A \rightarrow B$ function $a, a_1, a_2 \in A$, $b \in B$

f is **one-to-one**, or alternatively **injective**, if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$

f is **onto**, or **surjective**, if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$

f is **bijective** if it is both one-to-one and onto.

What can you say about f if ...

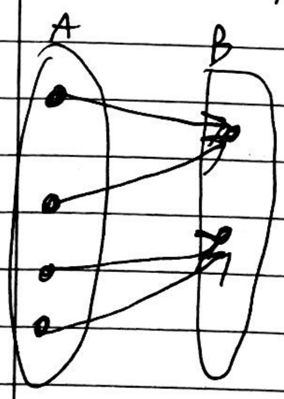
$\forall b \in B \ \exists! a \in A \ (f(a) = b)$

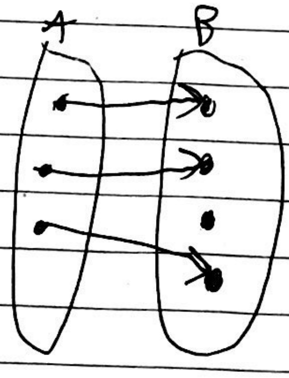
$\exists a_1 \in A \ \exists a_2 \in A \ (f(a_1) = f(a_2))$

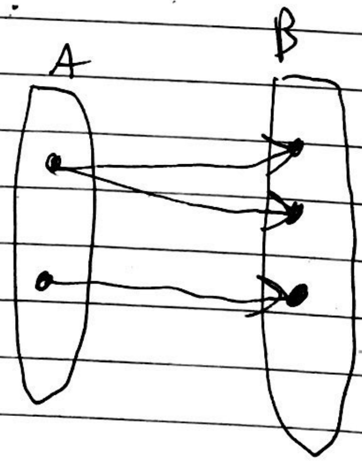
$\exists a_1 \in A \ \exists a_2 \in A \ (f(a_1) = f(a_2) \wedge a_1 \neq a_2)$

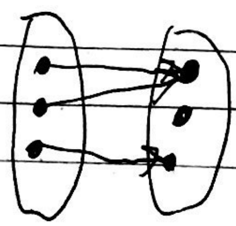
$\exists b \in B \ \forall a \in A \ (f(a) \neq b)$

One-to-one, onto, both, neither?









How to show a function $f: A \to B$ is ~~(blank)~~ ...

To show f is injective take arbitrary $a_1, a_2 \in A$ and assume $f(a_1) = f(a_2)$, then show $a_1 = a_2$

To show f is Not injective find elements $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and $a_1 \neq a_2$

To show f is surjective take $b \in B$ to be arbitrary, then find $a \in A$ so that $f(a) = b$.

To show f is not surjective find a particular $b \in B$ so that $f(a) \neq b$ for any $a \in A$.

Show $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = n^2$ is injective

Show $g: \mathbb{Z} \to \mathbb{R}$ given by $g(n) = n^2$ is not injective and not surjective.

Show that $h: \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = n + 2$ is bijective.

Erdős - Straus Conjecture

$\forall n \in \mathbb{Z} \quad n \geq 2 \quad \exists\, x, y, z \in \mathbb{Z}^+$ such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

Conjecture has been verified for $n \leq 10^{17}$ but remains an open problem.

Define a function $f: \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \longrightarrow \mathbb{Q}$ by letting $f((x, y, z)) = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$. Let $R = \{ f((x,y,z)) : (x,y,z) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+ \}$ denote the range.

The conjecture says $\frac{4}{n} \in R$ for each $n \in \mathbb{Z}, n \geq 2$.

Let $f: A \to B$ be a function so that $A, B \subseteq \mathbb{R}$.

f is called <u>increasing</u> if $x \leq y$ implies $f(x) \leq f(y)$ whenever $x, y \in A$.
(<u>Strictly increasing</u> replace $\leq$ by $<$)

f is called <u>decreasing</u> if $x \leq y$ implies $f(x) \geq f(y)$ whenever $x, y \in A$.
(<u>Strictly decreasing</u> similarly)

$f: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ $f(x) = x^2$ is strictly increasing

Pf) Take $x, y \in \mathbb{R}^+$ with $x < y$, then

$x^2 = x \cdot x < x \cdot y$ and $x \cdot y < y \cdot y = y^2$

so $x^2 < y^2$ and $f(x) < f(y)$ as desired.

$g: \mathbb{R} \longrightarrow \mathbb{R}$ $g(x) = x^2$ is not increasing

Pf) Take $-1 \leq 0$ both are elements of the domain $\mathbb{R}$. Then $f(-1) = 1 > 0 = f(0)$ so f is not increasing.

Show $h: \mathbb{R} \longrightarrow \mathbb{R}$ $h(x) = x^3$ is strictly increasing

Pf) Take $x, y \in \mathbb{R}$ with $x < y$

Case 1: $x = 0$

$f(x) = 0 < y^3 = f(y)$

Case 2: $y = 0$

$f(x) = x^3 < 0 = f(y)$

Case 3: $x < 0 < y$

$f(x) = x^3 < 0 < y^3 = f(y)$

Case 4: $0 < x < y$

$x < y$ $\quad x^2 < xy$ $\quad xy < y^2$

$x^3 < x^2 y$ $\quad x^2 y < x y^2$

$x^2 y < x \cdot y^2$ $\quad x y^2 < y^3$

$f(x) = x^3 < x^2 y < x y^2 < y^3 = f(y)$

case 5: $x < y < 0$