Natural Language

The problem with English $\rightsquigarrow$ logic is we are ofen not precise in English, but precision is essential in propositional logic.

Ex) A waiter tells you coffee or tea comes with your meal.

Really means exclusive or (maybe even nand), both coffee and tea will cost extra.

The book has further examples and discussion of the imprecision of natural language. I will not test you on ambiguous natural language.

Conditional Revisited

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

if p, then q
p only if q
q whenever p
q follows from p
$\vdots$

If you are riding the train, then you have paid the fare.
You are riding the train only if you have paid the fare.

1) If you score 100%, then you get an A
2) If you score $\geq 90\%$, then you get an A
3) You get an A only if you score 100%
4) You get an A only if you score $\geq 90\%$

1.3 Propositional Equivalences

A <u>tautology</u> is a compound proposition which is always true

A <u>contradiction</u> is a compound proposition which is always false

A <u>contingency</u> is a compound proposition which is neither a tautology nor a contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$

Propositions p + q are <u>logically equivalent</u> if $p \leftrightarrow q$ is a tautology. In this case we write $p \equiv q$.

Fill in the following table

De Morgan's Laws

$\neg(p \wedge q) \equiv$

$\neg(p \vee q) \equiv$

Notice analogy with distributive law

$(-1)(a+b) = (-a-b)$

More equivalences

$p \wedge T \equiv p$

$p \vee F \equiv p$

$p \vee T \equiv$

$p \wedge F \equiv$

$\neg(\neg p) \equiv$

$p \vee q \equiv q \vee p$

$p \wedge q \equiv q \wedge p$

$p \vee (q \vee r) \equiv (p \vee q) \vee r$

$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

$p \vee (q \wedge r) \equiv$

$p \wedge (p \vee r) \equiv$

$p \vee (p \wedge q) \equiv$

$p \wedge (p \vee q) \equiv$

$p \vee \neg p \equiv$

$p \wedge \neg p \equiv$

Find $\neg(p \to q)$ with logical equivalences, check answer with truth table. (Hint: $p \to q \equiv \neg p \vee q$)

p	q	$p \to q$	$\neg(p \to q)$	$p \wedge \neg q$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

$\neg(p \to q) \equiv \neg(\neg p \vee q)$

$\equiv \neg(\neg p) \wedge \neg q$

$\equiv p \wedge \neg q$

A compound proposition is <u>satisfiable</u> if it is a tautology or contingency (some evalution is true) otherwise it is <u>unsatisfiable</u>.

<u>Open Research problem</u>: Find a "good" algorithm to determine if a proposition is satisfiable. This is related to P vs. NP. (Alternative show no "good" algorithm can exist.)

Which propositions are satisfiable?

1) $(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg q)$

2) $(p \vee q \vee \neg r) \wedge (p \vee q \vee r)$

3) $(p \vee \neg q) \wedge (\neg p \vee q) \wedge (p \vee q) \wedge (\neg p \vee \neg q)$

p	q	r	$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg q)$
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	T

1.4 Predicates + Quantifiers

A <u>propositional function</u> is a declarative statement depending on one or more variables (with some domain) which is either true or false but not both for any choice of values for the variables

$P(x): x+2 \geq 6$ where x is a real number

$P(2)$ is a proposition "$4 \geq 6$" false
$P(7.5)$ is a proposition "$9.5 \geq 6$" true

$Q(x,y): x = y$ where x + y are integers

$Q(2,2)$ true
$Q(2,6)$ false

<u>Hoare logic</u> (1969)

A <u>Hoare triple</u> is $\{P\}\ C\ \{Q\}$ where P + Q are propositional functions + C is some code. These are use to verify correctness of a programs. In the book P is a "precondition", Q is a "postcondition" the term Hoare logic is not used

Ex) $\{x > 5\}\ \ y := x+2\ \ \{y > 7\}$

{a is an integer} $b := a/2$ {b is a rational number}

Using quantifiers turns a propositional function into a proposition.

$\forall x P(x)$ for all x $P(x)$

$\exists x P(x)$ there exists x such that $P(x)$

$\exists! x P(x)$ there exists a unique x such that $P(x)$

* Note the domain for x must be understood or specified

- Let the domain be the set of real numbers

$\forall x, x \geq 0$ is false $x=-3$ is a counterexample

$\forall x, x^2 \geq 0$ is true

$\exists x, x \geq 0$ is true

$\exists! x, x \geq 0$ is false $2 \geq 0$ and $7/2 \geq 0$

An element a so that $P(a)$ is false is called a counter-example to $\forall x P(x)$

How do we negate quantifiers?

$$\neg(\forall x P(x)) \equiv$$

$$\neg(\exists x P(x)) \equiv$$

Wason Selection task

P. Wason (1966) psychologist

< 10% correctly solved

Consider a deck of cards. One side has an integers while the otherside of each card has an A or B (exclusive or). Let the domain be the cards in the deck & consider

$\forall c$ (if c has an even number, then c has an "A")

$\boxed{4}$ $\boxed{7}$ $\boxed{B}$ $\boxed{A}$

Which of the four cards need to be flipped to check the proposition?

1.5 Nested Quantifiers

Let x & y be from the domain consisting of real numbers.

$\forall x \exists y (x+y=0)$

"for any x there is some y we can add to get zero"

True, take $y=-x$

what if we change the domain to positive real numbers?

How would we negate $\forall x \exists y (x+y=0)$?

$$\neg(\forall x \exists y (x+y=0)) \equiv \exists x \neg(\exists y (x+y=0))$$
$$\equiv \exists x \forall y \neg(x+y=0)$$
$$\equiv \exists x \forall y \; x+y \neq 0$$

The order of quantifiers matters.

Take the domain to be integers.

$\forall x \exists y \, (xy \geq 0)$

$\exists x \forall y \, (xy \geq 0)$

$\forall x \exists y \, (2x + y = 4)$

$\exists x \forall y \, (2x + y = 4)$

Determine the truth of each of the following for the case that
(a) The domain consists of real numbers
(b) The domain consists of integers

$\exists x \, (x^2 < x)$

$\forall x \, (2x \geq x)$

$\forall x \, (x^2 + 1 \geq 0)$

$\exists x \, (x^2 - 2 = 0)$

$\exists! x \, (x^2 - 2 = 0)$

$\exists x \forall y \, (x + y = y)$