1.6 Rules of inference

We now start to study mathematical proofs. We want to start from a <u>premise</u> (some given information) and make a <u>valid</u> argument to reach a conclusion.

The valid arugements will ~~b~~use tautologies from propositional logic.

$p \to q$	"If you have his phone #, then you can call"
p	"You have his phone #"
$\therefore q$	Therefore, "You can call"

This is a valid argument because
~~[illegible] make tautologies~~

p	q	$p \to q$	$((p \to q) \land p) \to q$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

$((p \to q) \land p) \to q$ is a tautology

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore q \end{array}$$

Modus pones

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore \neg p \end{array}$$

Modus tollens

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

Hypothetical syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore \end{array}$$

Disjunctive Syllogism

$$\begin{array}{l} p \\ \hline \therefore p \vee q \end{array}$$

addition

$$\begin{array}{l} p \wedge q \\ \hline \therefore p \end{array}$$

simplification

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array}$$

Conjunction

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore \end{array}$$

Resolution

Given the premises $(p \vee q) \to r$, $r \to s$, and p construct a valid argument to reach the conclusion s.

1. $(p \vee q) \to r$	Premise
2. $r \to s$	Premise
3. $(p \vee q) \to s$	Hypothetical syllogism
4. p	premise
5. $p \vee q$	addition
6. s	Hypothetical syllogism

Now lets add propositional functions + quantifiers.

Remember if $P(x)$ is a propositional function with domain D, then $P(d)$ is a proposition for any $d \in D$. Also $\forall x\, P(x)$ and $\exists x\, P(x)$ are propositions.

$$\frac{\forall x\, P(x)}{\therefore P(c) \text{ for any } c \in D} \qquad \frac{P(c) \text{ for } \boxed{\text{arbitrary}}\ c \in D}{\therefore \forall x\, P(x)}$$

$$\frac{\exists x\, P(x)}{\therefore P(c) \text{ for some } c \in D} \qquad \frac{P(c) \text{ for some } c \in D}{\therefore \exists x\, P(x)}$$

Let $D = \mathbb{R}$ (real numbers) and
$P(x)$: "$x^2 = x$"

Since $0^2 = 0$ (also $1^2 = 1$) we can take $c = 0$ and conclude $\exists x\, P(x)$.

Start with
"All students in the class know logic"
"If ~~someone~~ a student knows logic, then they are reasonable"
To conclude
George, a student in the class, is reasonable

1. $\forall s\, L(s)$
2. $L(\text{George})$
3. $\forall s\, (L(s) \to R(s))$
4. $L(\text{George}) \to R(\text{George})$
5. $R(\text{George})$

1.7 Intro to proofs

A **Theorem** is a important statement which has been shown to be true

A **lemma** is a helpful "small theorem" use to proof larger results.

A **corollary** is a statement that follows directly from a theorem.

A **Conjecture** is a statement thought to be true, but has not been proven.

(Mean Value Theorem)
If f is continuous on $[a,b]$ and differentiable on (a,b), then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

(Collatz Conjecture a.k.a $3n+1$ conjecture)

Let n be any positive integer.
If n is even, replace n by $\frac{n}{2}$
If n is odd, replace n by $3n+1$
Repeat

Conjecture: We eventually reach 1

Ex) 12, 6, 3, 10, 5, 16, 8, 4, 2, 1

Theorem are typically stated as $p \to q$ "if p, then q" sometime implicity it is meant $\forall x \, (P(x) \to Q(x))$

Thm) If n is an odd integer, then n^2 is odd

Really means $\forall n \, (P(n) \to Q(n))$ where the domain is all integer

$P(n)$: n is odd

$Q(n)$: n^2 is odd

Here n is <u>odd</u> if $n = 2k+1$ for an integer k.
Otherwise n is <u>even</u> and $n = 2k$.

Let try the proof our theorem.

Pf) Take an $\boxed{\text{arbitrary}}$ integer n. $\boxed{\text{Assume}}$ n is odd so that $n = 2k+1$ for some integer k.
In that case

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2+4k+1 \\ &= 2(2k^2+2k)+1 \end{aligned}$$

Since $2k^2+2k$ is an integer we see that n^2 is odd. Therefore the theorem is proven.

This is an example of a <u>direct proof</u>

Try proving the next theorem

Thm) If m & n are odd, then mn is odd.

Pf)

A real number r is <u>rational</u> if $r = \frac{a}{b}$ for integers a, b with $b \neq 0$. Otherwise r is <u>irrational</u>.

Thm) If x is irrational, then $\frac{1}{x}$ is irrational

Pf) We will show if $\frac{1}{x}$ is rational, then x is rational. This is the logically equivalent contrapositive. Assume $\frac{1}{x} = \frac{a}{b}$ is an arbitrary rational number. Here $a, b \neq 0$ are integers (why both $a, b \neq 0$?). Then $x = b/a$ is rational.

$\square$

This is a proof by contrapositive

Try proof by contrapositive

Thm) If n is an integer and $7n+4$ is even, then n is even.

Pf)