Try proof by contrapositive

Thm) If n is an integer and $7n+4$ is even, then n is even.

Pf)

Disproving existence + Proof by cases

Let x be from the domain of integers.

$\exists x\,(x^2 = 2)$

This is false, but how can be show it is false?

Prove the negation

$\neg(\exists x\,(x^2=2)) = \forall x\,(x^2 \neq 2)$

Proof by cases

case (i): $x=0$

If $x=0$, then $x^2=0 \neq 2$

case (ii): $x=\pm 1$

If $x=1$ or $x=-1$, then $x^2=1 \neq 2$

case (iii): $|x|>1$

If $|x|>1$, then $|x| \geq 2$ and $|x|^2 \geq 4$ so $x^2 \neq 2$.

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An *exhaustive proof* can be used when we only need to check finitely many cases.

Thm) If n is an integer and $0 \leq n \leq 3$, then $n^2 \leq n+10$.

Pf)

$$0^2 = 0 \leq 10 = 0+10$$
$$1^2 = 1 \leq 11 = 1+10$$
$$2^2 = 4 \leq 12 = 2+10$$
$$3^2 = 9 \leq 13 = 3+10$$

Prove there is no positive integer n such that $2n + n^2 = 36$.

Pf) First we reduce to use proof by exhaustion.

~~[illegible] disprove~~

Thm) n is a even integer if and only if n^2 is even

Suppose we want to prove a proposition p is true. We can do this by finding a contraction q such that $\neg p \to q$ is true. This is called <u>proof by contradiction</u>

$$\neg p \to q \equiv p \vee q \equiv p \vee F \equiv p$$

A common choice of contradiction is $q = r \wedge \neg r$

Thm) $\sqrt{2}$ is irrational

Pf) Assume $\sqrt{2} = \frac{a}{b}$ is rational. So a and b are integers with $b \neq 0$. Also assume a/b is in lowest terms (no common divisor)

$$2 = (\sqrt{2})^2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$$

$$2b^2 = a^2$$

This mean $a = 2k$ is even so $2b^2 = (a^2) = (2k)^2 = 4k^2$. Thus $b^2 = 2k^2$ so b is even.

$$q = \left(\frac{a}{b} \text{ lowest terms}\right) \wedge (a \text{ and } b \text{ both even})$$

$p = \sqrt{2}$ is irrational
$\neg p = \sqrt{2}$ ration
q is a contradiction

we have shown $\neg p \to q$

How does proof by contradiction work with statements $p \to q$? Remember $p \to q \equiv \neg(p \wedge \neg q)$

So, we can show $(p \wedge \neg q) \to F$, and proof of $p \to q$ by contradiction start as

"assume p and $\neg q$ ..."

Prove: If n^3+2 is even, then n is even.

Pf) Assume n^3+2 is even and n is odd

$n^3+2 = 2k$ for an integer k
$n = 2\ell+1$ for an integer ℓ

$$\begin{aligned} n^3+2 &= (2\ell+1)^3+2 \\ &= 8\ell^3+12\ell^2+6\ell+1+2 \\ &= 2(4\ell^3+6\ell^2+3\ell+1)+1 \end{aligned}$$

$$2k = 2(4\ell^3+6\ell^2+3\ell+1)+1$$

$$k = (4\ell^3+6\ell^2+3\ell+1)+\tfrac{1}{2} \leftarrow \text{NOT an integer}$$

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