Chapter 2 Basic structures: Sets, Functions, Sequences, Sum, and Matrices

2.1 Sets

A set is a collection of objects called members or elements. Neither the order nor repetition of elements matter.

We write $a \in A$ if a is an element of the set A. We write $a \notin A$ if a is not an element of the set A.

Examples

$A = \{1,2,3,4,5\} = \{1,3,5,2,4\}$ $\quad 2 \in A$, $10 \notin A$

$\{9, x, 6\} = \{9, x, x, 6\}$

$\mathbb{N} = \{0,1,2,3,\dots\}$ Natural numbers

$\mathbb{Z} = \{\dots,-2,-1,0,1,2,\dots\}$ Integers

$\mathbb{Z}^+ = \{1,2,3,\dots\}$ Positive integers

$\mathbb{Q} = \{p/q : p,q \in \mathbb{Z}\ q \neq 0\}$ Rational number

$= \{p/q \mid p,q \in \mathbb{Z}\ q \neq 0\}$

$\mathbb{R}$ Real numbers

$\mathbb{R}^+$ Positive real numbers

$\mathbb{C}$ Complex numbers

$\{x : x \text{ has property } P\}$ is called set builder notation.

$\{x \in \mathbb{Z} : x = 2k+1 \text{ for } k \in \mathbb{Z}\}$ ← odd integers

$\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$

Describe the following sets

$\{x \in \mathbb{R} : x^2 = 100\} = \{-10, 10\}$

$\{x \in \mathbb{R}^+ : x^2 = 100\} = \{10\}$

The empty set is $\phi = \{\}$ which is the special set with no elements.

Cultural aside

We are doing naive set theory. This will be sufficient for our purpose, but there are issues with this theory.

In the 19th Century Cantor and Dedekind ~~[illegible]~~ initiated the study of set theory. Today it is still research ~~and~~ is part of the foundations of mathematics.

There is a taste of set theory called "Russell's paradox" (1901) (also Zermelo 1899)

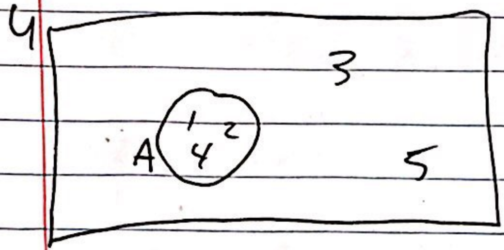
Let $R = \{S : S \text{ is a set and } S \notin S\}$

Assume $R \in R$, then $R \notin R$

Assume $R \notin R$, then $R \in R$

we can use Venn diagrams to represent sets. We let U be some <u>universal set</u> including all objects under consideration.

$U = \{1, 2, 3, 4, 5\}$ $A = \{1, 2, 4\}$



A is a <u>subset</u> of B and we write $A \subseteq B$ if $\forall x \in U,\ x \in A \longrightarrow x \in B$. That is, every element of A is a element of B.

In this case B is a <u>superset</u> of A. we may write eiter $A \subseteq B$ or $B \supseteq A$.

Thm) For any set S, $\phi \subseteq S$ and $S \subseteq U$

Example) How to show $A \supseteq B$.

Let $A = \{x^2 : x \in \mathbb{Z}\}$, $B = \{x^4 : x \in \mathbb{Z}\}$
$A = \{1, 4, 9, \cdots\}$, $B = \{1, 16, 81, \ldots\}$

Choose $b \in B$ arbitrarily, then $b = x^4$ for some $x \in \mathbb{Z}$. But then $b = (x^2)^2$ and $x^2 \in \mathbb{Z}$. So, $b \in A$ and it follows $A \supseteq B$.

* To show $A = B$, show both $A \subseteq B$ and $A \supseteq B$.

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If a set S has exactly n distinct elements where n is a nonnegative integer, then S is a <u>finite</u> set. In this case we say S has <u>cardinality</u> n and write $|S| = n$.

Otherwise S is said to be <u>infinite</u>.

$$|\phi| = 0$$

$$|\{x \in \mathbb{Z} : x^2 \leq 16\}| = 9 \quad \{-4, -3, -2, -1, 0, \cdots, 4$$

$$|\{x \in \mathbb{Z} : |x| < 3/2\}| = 3 \quad \{-1, 0, 1\}$$

$$|\{n \in \mathbb{N} : n^2 = n\}| = 2 \quad \{0, 1\}$$

<u>Cultural facts</u>

Cardinality can be defined for infinite sets (but we won't do this).

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$$

The <u>powerset</u> of a set S is denoted $\wp(S)$ and is the set of all subsets of S

$$\wp(\{1,2\}) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$$

$$\wp(\{1,2,3\}) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}$$

The Cartesian product of sets A and B is

$$A \times B = \{(a,b) : a \in A, b \in B\}$$

Examples

$\{1,2\} \times \{3,4\} = \{(1,3),(1,4),(2,3),(2,4)\}$

$\{a,b,c\} \times \{x,y\} = \{(a,x),(a,y),(b,x),(b,y),$
$(c,x),(c,y)\}$

Here $(a,b) \in A \times B$ is called an ordered pair.
An ordered n-tuple $(a_1, a_2, \ldots, a_n)$ show up
to Cartesian products of n sets

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for } 1 \le i \le n\}$$

$\{1\} \times \{1,2\} \times \{a,b\} = \{(1,1,a),(1,1,b),(1,2,a),(1,2,b)\}$

What is the cardinality of powersets and
Cartesian products of finite sets

$$|P(A)| = 2^{|A|}$$

$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdots |A_n|$$