1 The Foundations: Logic & Proofs

1.1 Propositional Logic

A <u>proposition</u> is a declaritive sentence which is either true or false but not both

- Toronto is in Europe. — Yes a proposition
- What's for lunch? — No
- $x + 5 = 3/4$ — No
- $2 = 6$ — Yes
- $1 + 3 = 4$ — Yes
- Toronto is in Canada. — Yes

$p, q, r, \ldots$ will be used to denote propositions

How can propositions be combined to create another propositions? What operations can be used?

<u>negation</u>	$\neg p$	"not p"
<u>conjunction</u>	$p \wedge q$	"p and q"
<u>disjunction</u>	$p \vee q$	"p or q"

When is $\neg p$ true?
When is $p \wedge q$ true?
When is $p \vee q$ true?

Truth table

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

- $\neg p$ is true when p is false.
- $p \wedge q$ is true when both p & q are true.
- $p \vee q$ is true when at least one of p or q is true

exclusive or (a.k.a. XOR) is denote by $p \oplus q$ and is true when exactly one of p or q is true and is false otherwise

Try to express $p \oplus q$ in terms of $\neg$, $\wedge$, and $\vee$.

$(p \wedge \neg q) \vee (\neg p \wedge q)$

p	q	$p \wedge \neg q$	$\neg p \wedge q$	$(p \wedge \neg q) \vee (\neg p \wedge q)$	$p \oplus q$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	F	T	T	T
F	F	F	F	F	F

Let p, q be propositions. The <u>conditional statement</u> $p \rightarrow q$ is the proposition which is true except in the case that p is true and q is false.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

"if p, then q"
"p implies q"
"q follows from p"

"q is neccessary for p"
"p is sufficient for q"
and others...

p — q

- You ~~can~~ are riding the train only if you have paid the fare.

This a conditional statement
  p = You can ride the train
  q = you have paid the fare

$$p \rightarrow q$$

Try to express $p \rightarrow q$ using $\neg, \vee, \wedge$

~~[illegible]~~q $\neg(p \wedge \neg q)$ $(p \wedge q) \vee \neg p$

$\neg p \vee q$

denoted $\equiv$

Two propositions are called <u>equivalent</u> if the always have the same truth values.

The <u>converse</u> of $p \to q$ is $q \to p$

The <u>contrapositive</u> of $p \to q$ is $\neg q \to \neg p$

The <u>inverse</u> of $p \to q$ is $\neg p \to \neg q$

Is $p \to q$ equivalent to any of its converse, contrapositives or inverse?

equivalent to contrapositive

method 1: Truth tables

method 2: $p \to q \equiv (\neg p \vee q)$

$\neg q \to \neg p \equiv (\neg(\neg q) \vee \neg p)$

"if and only if"

The biconditional statement $p \leftrightarrow q$ is the proposition which is true exactly when p & q have the same truth values.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Express $p \leftrightarrow q$ in terms of $\neg, \vee, \wedge, \rightarrow$.
That is find equivalent propositions to $p \rightarrow q$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$

The precadance of logical operators is

1) $\neg$
2) $\wedge$
3) $\vee$
4) $\rightarrow$
5) $\leftrightarrow$

$\neg p \wedge q$ is $(\neg p) \wedge q$

$p \vee q \wedge r$ is $p \vee (q \wedge r)$

$p \vee q \rightarrow r \wedge s$ is $(p \vee q) \rightarrow (r \wedge s)$