READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything excepts pens, pencils and erasers.
- If you need scratch paper, the last page is blank.
- **Show all your work unless otherwise indicated.** Write your answers clearly! Include enough steps for the grader to be able to follow your work.
- You will be given exactly 60 minutes for this exam.

I have read and understand the above instructions: ______________________________

SIGNATURE

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (5 points) The set of all odd integers is:
   - **A. countably infinite.**
   - B. uncountably infinite.
   - C. finite.

2. (5 points) The function $f(x) = 5x^2 + 3\sqrt{x}$ is:
   - A. $O(x)$
   - **B.** $O(x^2)$
   - C. $O(\sqrt{x})$

3. (5 points) If n is any integer, then:
   - A. $\lfloor -7.5 + n \rfloor = -8$
   - B. $\lfloor -7.5 + n \rfloor = -7 + n$
   - **C.** $\lfloor -7.5 + n \rfloor = -8 + n$

Extra Work Space.

Fill in the Blanks. No work needed. No partial credit available.

4. (5 points) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x - 5$ is a bijection and its inverse is

$$f^{-1}(x) = \underline{\ \frac{y+5}{3}\ \qquad\qquad}.$$

5. (5 points) If the sequence L_n is defined by $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$ then $L_5 =$ __11________.

6. (5 points) The value of the sum $\sum_{i=0}^{3} 3^i$ is __40________.

Extra Work Space.

Standard Response Questions. Show all work to receive credit.

7. (20 points) Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f(a, b) = 2a + 3b$. Show that f is surjective but not injective.

Solution: To show that f is surjective we choose an arbitrary element $n \in \mathbb{Z}$. If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$ and in that case $f(k, 0) = 2k = n$. If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$ and in that case $f(k - 1, 1) = 2(k - 1) + 3 = 2k + 1 = n$. Therefore we can conclude that f is surjective.

To show that f is not injective we take the distinct elements $(3, 0), (0, 2) \in \mathbb{Z} \times \mathbb{Z}$ and observe that $f(3, 0) = 6 = f(0, 2)$.

8. (15 points) Show that the function $f : \mathbb{N} \to \mathbb{Z}^+$ defined by $f(n) = n!$ is not $O(n^2 - n)$.

Solution: Assume for a contradiction by $f(n)$ was $O(n^2 - n)$. Then there exits a constant C and $N \in \mathbb{N}$ so that $n! \leq C(n^2 - n)$ for all $n \geq N$. If $n! \leq C(n^2 - n)$ for all $n \geq N$, then

$$\begin{aligned} n! &\leq C(n^2 - n) \\ \frac{n!}{n^2 - n} &\leq C \\ \frac{n(n-1)(n-2)\cdots 1}{n(n-1)} &\leq C \\ (n-2)! &\leq C \end{aligned}$$

and $(n-2)! \leq C$ for all $n \geq N$. This is a contradiction and therefore we can conclude that $f(n)$ is not $O(n^2 - n)$. Since such a constant C must be nonnegative for an $n > C + 2$ we have $(n-2)! > C$.

9. (15 points) Recall that Hilbert's Grand Hotel is a hotel with a countably infinite number of rooms. Assume that every room of Hilbert's Grand Hotel is occupied. Suppose that two buses each carrying a countably infinite number of new guests arrive. Explain how all the new guests from the two buses can be accommodated in Hilbert's Grand Hotel without evicting any of the current guests.

Solution: The countably infinite rooms in Hilbert's Grand Hotel are numbered $1, 2, 3, \ldots$ with positive integers. First let us move each current guest in room n to room $3n$. Next, for each $k = 1, 2, 3, \ldots$ let us put the kth guest from the first bus in room $3k + 1$. Finally for each $k = 1, 2, 3, \ldots$ we assign the kth guest from the second bus to room $3k + 2$. Now all the guests have a room (and we actually have extra space since room 1 and room 2 are not occupied).

10. (20 points) Prove using induction that

$$\sum_{k=1}^{n}(3k-2) = \frac{n(3n-1)}{2}$$

for any integer $n \geq 1$.

Solution: We first check the base case and see that

$$\sum_{k=1}^{1}(3k-2) = 1 = \frac{(1)(2)}{2}.$$

Next we inductively assume that

$$\sum_{k=1}^{n}(3k-2) = \frac{n(3n-1)}{2}$$

for sum given $n \geq 1$. We then compute

$$\begin{aligned}
\sum_{k=1}^{n+1}(3k-2) &= \sum_{k=1}^{n}(3k-2) + (3(n+1)-2) \\
&= \frac{n(3n-1)}{2} + 3n + 1 \\
&= \frac{n(3n-1)}{2} + \frac{6n+2}{2} \\
&= \frac{3n^2 - n + 6n + 2}{2} \\
&= \frac{3n^2 + 3n + 2n + 2}{2} \\
&= \frac{3n(n+1) + 2(n+1)}{2} \\
&= \frac{(n+1)(3n+2)}{2} \\
&= \frac{(n+1)(3(n+1)-1)}{2}
\end{aligned}$$

and conclude the result in true by induction.

Extra Work Space.