Name: ______________________________

YorkU email: ______________________________

Student Number: ____________________

READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything excepts pens, pencils and erasers.
- If you need scratch paper, the last page is blank.
- **Show all your work unless otherwise indicated.** Write your answers clearly! Include enough steps for the grader to be able to follow your work.
- You will be given exactly 60 minutes for this exam.

I have read and understand the above instructions: ______________________________

SIGNATURE

Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. (5 points) Which of are the following relations on $\{1, 2, 3\}$ is symmetric?

   A. $\{(1, 2), (2, 3), (2, 1), (3, 2)\}$

   B. $\{(1, 1), (2, 2), (3, 3), (1, 3)\}$

   C. $\{(1, 2), (2, 3), (3, 1)\}$

2. (5 points) The graph $G = (V, E)$ where

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}, \{2, 4\}\}$$

   and $V = \{1, 2, 3, 4\}$ is:

   A. a bipartite graph

   B. NOT a bipartite graph

3. (5 points) If a_n is the number of bit strings which do not contain four consecutive 0's, then a_n satisfies which recurrence:

   A. $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$

   B. $a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}$

   C. $a_n = 2a_{n-1} + 2^{n-1}$

   D. $a_n = 3a_{n-2} + a_{n-4}$.

Extra Work Space.

Fill in the Blanks. No work needed. No partial credit available.

4. (5 points) If $f(1) = 3$ and $f(n) = f(n/5) + 1$, then $f(125) =$ 6.

5. (5 points) Give an example of a partition of the set $\{1, 2, x, y, z\}$:

$\{\{1, x\}, \{2, y, z\}\}$

6. (5 points) The number of edges in K_5 (the complete graph on 5 vertices) is 10.

Extra Work Space.

Standard Response Questions. Show all work to receive credit.

7. (15 points) Consider the set Σ^* of bit strings (i.e. $\Sigma = \{0, 1\}$). Let $A \subseteq \Sigma^*$ be the set of bit strings such that any 0 is followed by at least two 1's. Give a recursive definition of the set A and list all bit strings of length 5 that are in A.

Solution: We can define the set A by the following properties:

- $\lambda \in A$
- $w1 \in A$ if $w \in A$
- $w011 \in A$ if $w \in A$

The bit strings of length 5 in A are:

$$01111, 10111, 11011, 11111$$

8. (20 points) Solve the recurrence relation $b_n = \frac{7}{2}b_{n-1} - \frac{3}{2}b_{n-2}$ with $b_0 = 2$ and $b_1 = 3$.

Solution: The characteristic equation is

$$r^2 - \frac{7}{2}r + \frac{3}{2} = (r - \frac{1}{2})(r-3)$$

and so

$$b_n = A\left(\frac{1}{2}\right)^n + B(3)^n$$

for some constants A and B. The initial conditions give us

$$\begin{aligned} A + B &= 2 \\ \frac{1}{2}A + 3B &= 3 \end{aligned}$$

which implies $-5B = -4$. So, we see that $B = 4/5$ and $A = 6/5$. Therefore

$$b_n = \frac{6}{5}\left(\frac{1}{2}\right)^n + \frac{4}{5}(3)^n.$$

9. (15 points) We consider the relation

$$R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : a - b = 5k \text{ for some } k \in \mathbb{Z}\}$$

on $\mathbb{Z}$. Prove that R is an equivalence relation.

Solution: For any $a \in \mathbb{Z}$ we have that

$$a - a = 0 = 5 \cdot 0.$$

Thus, $(a, a) \in R$ and R is reflexive.

Assume that $(a, b) \in R$, then $a - b = 5k$ for some $k \in \mathbb{Z}$. Then $b - a = -(5k) = 5(-k)$ and so $(b, a) \in R$ since $-k \in \mathbb{Z}$. Hence, R is symmetric.

Assume that $(a, b) \in R$ and that $(b, c) \in R$. This means that

$$a - b = 5k_1$$
$$b - c = 5k_2$$

for some $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$. We then find

$$a - c = (a - b) + (b - c) = 5k_1 + 5k_2 = 5(k_1 + k_2)$$

and so $(a, c) \in R$ because $k_1 + k_2 \in \mathbb{Z}$. Therefore R is transitive and hence an equivalence relation.

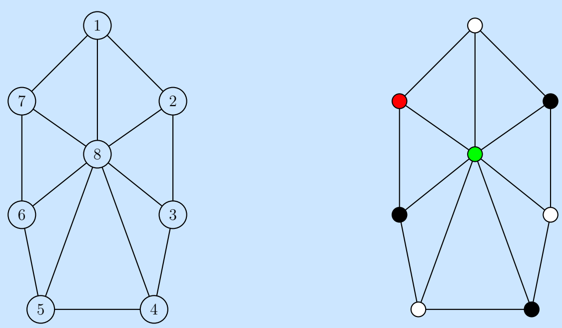
10. (20 points) Let W_n be the graph with vertex set $V = \{1, 2, \ldots, n+1\}$ and edge set

$$E = \{\{i, n+1\} : 1 \leq i \leq n\} \cup \{\{i, i+1\} : 1 \leq i \leq n-1\} \cup \{\{1, n\}\}.$$

(a) Draw the graph W_7.

(b) Find the chromatic number of W_7.

Solution: The graph W_7 looks like



on the left with vertices labeled and on the right with a proper coloring with 4 colors. Since the vertices $\{1, 2, 3, 4, 5, 6, 7\}$ make an odd length cycle these vertices need at least 3 colors. Since the vertex 8 is connected to all other vertices it needs to be its own color. Therefore we can conclude the chromatic number of W_7 is 4

Extra Work Space.