Hyperfine structure and Zeeman splitting in two-fermion bound-state systems

Andrei G. Terekidi,* Jurij W. Darewych,† and Marko Horbatsch‡
Department of Physics and Astronomy, York University, Toronto, Ontario, Canada M3J 1P3
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A relativistic wave equation for bound states of two fermions with arbitrary masses, which are exposed to a magnetic field, is derived from quantum electrodynamics. The interaction kernels are based upon the generalized invariant $M$-matrices for interfermion and fermion-field interactions. As an application we calculate the energy corrections in a weak homogeneous $B$ field to obtain the Zeeman splitting of the hyperfine structure and $g$ factors in the lowest order [i.e., to $O(a^0)$]. Landé $g$ factors are presented for several of the first excited states of hydrogen, muonium, and muonic hydrogen.

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I. INTRODUCTION

The relativistic treatment of energy levels of two-fermion atomic systems (including atomic hydrogen, hydrogenlike ions, helium-3 ion, muonium, and muonic hydrogen), as well as their fine structure (FS) and hyperfine structure (HFS) in an external uniform magnetic field (Zeeman effect), is an important problem. The theoretical knowledge of energy spectra and transition frequencies provides a test of two-body bound-state QED [1]. One can then obtain information about the character of the coupling in the system, the g- factors in the lowest order [2–5], and fundamental physical constants such as the Rydberg constant $R_n$, and the fine structure constant $\alpha$ [6]. The Zeeman effect in the HFS can be used as a diagnostic tool for solar photospheric magnetic fields [7], fusion research and plasma physics, where the magnetic field is applied to control the shape and position of the plasma [8].

In the lowest-order approximation the linearly dependent part of the energy splitting for a two-fermion system placed in a weak static magnetic field $B$ can be written as [1,9–11]

$$\Delta E^{ext}_{F,m_F,j_1,F,F} = (\mu_{B1} g_1 + \mu_{B2} g_2)B m_F,$$

where $F, m_F, j_1, \ell, s_1, I$ are quantum numbers, which characterize the system: $s_1$ and $I$ are the spins of the first and second particle, respectively, $\ell$ and $j_1$ represent the orbital and total angular momentum quantum numbers of the first particle. The total angular momentum of the system is denoted by the quantum number $F = j_1 + I, j_1 + I - 1, \ldots, |j_1 - I|$. The projection of the total angular momentum on the $B$ direction is $m_F = -F, -F + 1, \ldots, -1, F$. The “Bohr magnetons” for the two particles are defined as $\mu_{B1} = Q_1 \hbar / 2 m_{1c}$ and $\mu_{B2} = -Q_2 \hbar / 2 m_{2c}$, where $Q_1, Q_2 > 0$. Usually, in our notation $m_1$ and $m_2$ correspond to the light and heavy particle, respectively. Assuming that the energy-level splitting (1) is much smaller then the HFS splitting, $\Delta E^{ext} \ll \Delta E_{HFS}$, the Landé ($g$-) factors $g_1$ and $g_2$ take the form [9–11]

$$g_1 = g_{j_1} = \frac{F(F+1) + j_1(j_1+1) - I(I+1)}{2F(F+1)},$$

where

$$g_{j_1} = 1 + (gs_{s_1} - 1) \frac{j_1(j_1 + 1) + s_1(s_1 + 1) - \ell(\ell + 1)}{2j_1(j_1 + 1)},$$

and

$$g_2 = g_{s_2} = \frac{F(F+1) - j_1(j_1 + 1) + I(I+1)}{2F(F+1)}.$$
reformulation of QED and the variational Hamiltonian formalism developed earlier [18–20]. A relativistic two-fermion wave equation for arbitrary fermion masses is, thus, derived from first principles. A solution of this equation permits, in principle, to obtain all QED energy corrections to any order of the coupling constant [18]. In the present paper we extend the method to derive the integral wave equation in momentum space for the case where a uniform weak magnetic field is present. We calculate the Zeeman splitting of the HFS energy levels to $O(a^2)$ for all quantum states and unrestricted values for the fermion masses. We obtain a different result for the $g$ factor, Eqs. (38)–(41), and demonstrate that it coincides with Eqs. (2) and (4) in the case of $m_2 \gg m_1$, as long as the intrinsic moment of $m_1$ is restricted to the Dirac value $\gamma = 2$.

The modification of the wave equations due to the external magnetic field is presented in Sec. II. In Sec. III we provide the classification of the quantum states, and a partial-wave decomposition of the momentum-space equations. Section IV contains expressions for the Zeeman energy splittings of the HFS levels, and the $g$-factor results. Numerical values for the Landé factors are compared with data from Eqs. (18) and (4) for various excited states of hydrogen, muonium, and muonic hydrogen. In most expressions we use natural units $\hbar = c = 1$.

II. BOUND-STATE VARIATIONAL WAVE EQUATION

For two-fermion systems without external fields wave equations were derived in Refs. [18,19] on the basis of a modified QED Lagrangian [21,22]. With this Lagrangian a simple Fock-space trial state

$$|\psi_{\text{trial}}\rangle = \sum_{s_{12}} \int \frac{d^3p_1d^3p_2}{\sqrt{(2\pi)^6}} F_{s_{12}}(p_1,p_2) b_{p_1}^\dagger D_{p_2}^\dagger |0\rangle,$$

(6)
sufficed to obtain HFS levels correct to fourth order in the fine-structure constant. Here $b_{p_1}^\dagger$ and $D_{p_2}^\dagger$ are creation operators for a free fermion of mass $m_1$ and an (anti-)fermion of mass $m_2$, respectively, and $|0\rangle$ is the trial vacuum state such that $b_{p_1}^\dagger |0\rangle = D_{p_2}^\dagger |0\rangle = 0$.

As discussed in Sec. III below, the four adjustable functions $F_{s_{12}}$ must be chosen so that the trial state (6) is an eigenstate of the relativistic total angular momentum operator, its projection, and parity (as well as charge conjugation for the case $m_1 = m_2$ such as positronium).

A variational principle is invoked to obtain a momentum-space wave equation for the amplitudes [18]

$$0 = \sum_{s_{12}} \int \frac{d^3p_1d^3p_2}{(2\pi)^3} (\omega_{p_1} + \omega_{p_2} - E) F_{s_{12}}(p_1,p_2) \delta F_{s_{12}}(p_1,p_2)$$

$$- \frac{m_1m_2}{2} \sum_{s_{12}1\sigma_1\sigma_2} \int \frac{d^3d^3q}{\sqrt{\omega_{p_1}\omega_{p_2}\omega_{q_1}\omega_{q_2}}} F_{s_{12}}(q_1,q_2)$$

$$\times (-i) \tilde{M}_{s_{12}\sigma_1\sigma_2}(p_1,p_2,q_1,q_2) \delta F_{s_{12}}(p_1,p_2),$$

(7)

where $\omega_{p_1} = p_1^2 + m_1^2$ and $\omega_{p_2} = p_2^2 + m_2^2$. The interaction is governed by the generalized invariant matrix $\tilde{M}_{s_{12}\sigma_1\sigma_2}(p_1,p_2,q_1,q_2)$. It has the form

$$\tilde{M}_{s_{12}\sigma_1\sigma_2}^{(1)}(p_1,p_2,q_1,q_2) = \tilde{M}_{s_{12}\sigma_1\sigma_2}^{\text{opt}}(p_1,p_2,q_1,q_2)$$

$$+ \tilde{M}_{s_{12}\sigma_1\sigma_2}^{\text{ext}}(p_1,p_2,q_1,q_2),$$

(8)

where $\tilde{M}_{s_{12}\sigma_1\sigma_2}^{\text{opt}}(p_1,p_2,q_1,q_2)$ is the usual invariant matrix element, corresponding to the one-photon exchange Feynman diagram [19,20].

The element $\tilde{M}_{s_{12}\sigma_1\sigma_2}^{\text{ext}}(p_1,p_2,q_1,q_2)$ represents the interaction with a given external classical field $A_{\mu}^{\text{ext}}$,

$$\tilde{M}_{s_{12}\sigma_1\sigma_2}^{\text{ext}}(p_1,p_2,q_1,q_2) = i(2\pi)^{3/2}$$

$$\frac{\sqrt{\Omega_{p_1}\Omega_{p_2}}}{m_2} A_{\mu}^{\text{ext}}(p_1 - q_1)i\tilde{u}(p_1,s_1)(-i\sigma_1)\gamma^\mu\tilde{u}(q_1,\sigma_1)\delta_{s_2,s_2}^\dagger(p_2 - q_2)$$

$$+ \frac{\sqrt{\omega_{p_1}\omega_{q_1}}}{m_1} A_{\mu}^{\text{ext}}(q_2 - p_2)\tilde{V}(p_2,\sigma_2)(-i\sigma_2)\gamma^\mu V(q_2,s_2)\delta_{s_1,s_1}^\dagger(p_1 - q_1).$$

(9)

The ansatz (6) cannot accommodate processes that include the emission or absorption of real, physical (as opposed to virtual) photons. Such radiative processes could be included by generalizing the trial state. Here we limit ourselves to the form (6), i.e., the effects of radiative decay or absorption of radiation are ignored in the present work.

In order to obtain the Landé factors we evaluate the $\tilde{M}_{s_{12}\sigma_1\sigma_2}^{\text{ext}}$ matrix (9) in a stationary uniform magnetic field $B = Bz$. The vector potential can be chosen as

$$A_1^{\text{ext}}(x) = -\frac{1}{2}\gamma B, \quad A_2^{\text{ext}}(x) = \frac{1}{2}\gamma B, \quad A_0^{\text{ext}}(x) = A_3^{\text{ext}}(x) = 0.$$

(10)

The inverse Fourier transform of the nonzero components yields
\[ A_1^{c}\text{st}(k) = \frac{(2\pi)^{3/2}}{2} \frac{d\delta(k_c)}{dk} \delta(k), \]

\[ A_2^{c}\text{st}(k) = -\frac{(2\pi)^{3/2} i B}{2} \frac{d\delta(k_c)}{dk} \delta(k). \]

Using the semirelativistic expansion

\[ \mathcal{M}_{s_1s_2\sigma_1\sigma_2}^{c\text{st}}(p_1, p_2; q_1, q_2) = \frac{(2\pi)^{3/2}}{2c} \left( \frac{Q_1}{m_1} A_j^{c\text{st}}(p_1 - q_1) \varphi_{\sigma_1}^i \left( \gamma_5 \gamma_5 \cdot (p_1 - q_1) \right) + q_1 + p_1 \right) \varphi_{\sigma_1} \delta_{\sigma_2\sigma_2} \delta(p_2 - q_2) \right), \]

where \( \chi_1^j = [0, 1], \) \( \chi_2^j = [-1, 0], \) and \( j = 1, 2. \) It is straightforward to show that

\[ (q_1) A_j^{c\text{st}}(p_1 - q_1) = -\frac{(2\pi)^{3/2} B}{2} \hat{L}_{1z}(q_1) \delta^{i}(p_1 - q_1), \]

and

\[ A_j^{c\text{st}}(p_1 - q_1) \varphi_{\sigma_1}^i \left( \gamma_5 \gamma_5 \cdot (p_1 - q_1) \right) + q_1 + p_1 \right) \varphi_{\sigma_1} = -\frac{(2\pi)^{3/2} B}{2} \varphi_{\sigma_1} \hat{L}_{1z}\varphi_{\sigma_1} \delta^{i}(p_1 - q_1). \]

where \( \hat{L}_{1z}(q_1) \) is the \( z \) component of the angular momentum operator of the particle with mass \( m_1, \)

\[ \hat{L}_{1z}(q_1) = -i \left( q_{1x} \frac{\partial}{\partial q_{1y}} - q_{1y} \frac{\partial}{\partial q_{1x}} \right). \]

Taking \( \varphi_{\sigma_1} \) to be the eigenstates of the spin operator \( \hat{S}_{1z} = \frac{1}{2} \hat{S}_{1z}, \) and using a similar procedure for the second particle, we obtain

\[ \mathcal{M}_{s_1s_2\sigma_1\sigma_2}^{c\text{st}}(p_1, p_2; q_1, q_2) = -\frac{(2\pi)^{3/2} B}{2c} \left( \frac{Q_1}{m_1} [2\varphi_{\sigma_1} \hat{S}_{1z} \varphi_{\sigma_1} + \delta_{\sigma_1\sigma_2} \hat{L}_{1z}(q_1) \delta^{i}(p_1 - q_1) \delta^{i}(p_2 - q_2) \right), \]

or

\[ \mathcal{M}_{s_1s_2\sigma_1\sigma_2}^{c\text{st}}(p_1, p_2; q_1, q_2) = -\frac{(2\pi)^{3/2} B}{2c} \left( \frac{Q_1}{m_1} [2\varphi_{\sigma_1} \hat{S}_{1z} \varphi_{\sigma_1} + \delta_{\sigma_1\sigma_2} \hat{L}_{1z}(q_1) \delta^{i}(p_1 - q_1) \delta^{i}(p_2 - q_2) \right), \]

where the spin projection quantum numbers \( \tilde{m}_\sigma \) can take the values \( \pm 1/2. \) The quantities \( \mu_{B1} \) and \( \mu_{B2} \) are the “Bohr magnetons” defined in the previous section. As expected, a unit of spin interacts with a magnetic field twice as strongly as a unit of orbital angular momentum.

By going to the next order in the expansion of the invariant \( \mathcal{M} \) matrix one can obtain self-energy corrections, which lead to divergent loop integrals that have to be cured by charge renormalization. The vertex term modifies the Dirac value of the magnetic moment by a factor \( (1 + k), \) where \( k \) is the anomaly (Schwinger correction). This factor can be included in our calculation by a replacement \( 2\tilde{m}_{\sigma_1} \) and \( 2\tilde{m}_{\sigma_2} \) in Eq. (18) by \( g_{s_1} \tilde{m}_{\sigma_1} \) and \( g_{s_2} \tilde{m}_{\sigma_2} \), respectively, where \( g_{s_1,2} = 1 + k_{1,2} \). The anomaly is the lowest-order QED correction to the \( g \) factor \( \Delta_{s_12}^{QED} = 2k_{1,2} \) in Eq. (5).

### III. PARTIAL-WAVE DECOMPOSITION AND RADIAL WAVE EQUATIONS

The present work is an extension of Ref. [18], in which the partial-wave decomposition of the wave equation has been provided. The external magnetic field is treated as a first-order perturbation, which implies that the quantum labels for the eigenstates do not change. The restrictions on the magnetic field strength to justify a perturbative treatment of Eq. (18) are

\[ B \leq \min \left[ \frac{\alpha^4 m_e c^2}{\mu_{B1}}, \frac{\alpha^4 m_e c^2}{\mu_{B2}} \right], \]

where \( \alpha = Q_1 Q_2 / 4\pi, \) and \( m_e = m_1 m_2 / (m_1 + m_2) \) is the reduced mass. A more explicit restriction on \( B \) will be presented in Sec. IV.
As outlined in Ref. [18] the trial state (6) is taken to be an eigenstate of total linear momentum $\hat{P}$, total angular momentum squared $\hat{J}$, its projection $J_z$, and parity $\hat{P}$. It is natural to work in the rest frame, where the total linear momentum vanishes. In this frame the adjustable functions take the form

$$F_{s_1s_2}(p_1, p_2) = \delta(p_1 + p_2) F_{s_1s_2}(p_1),$$

where $F_{s_1s_2}(p_1)$ (using $p_1 = p$) can be written as

$$F_{s_1s_2}(p) = \sum_{\ell_1s_1} \sum_{m_1s_1} \sum_{\ell_2s_2} \sum_{m_2s_2} f_{\ell_1s_1m_1s_1}(p) Y_{m_1s_1}^{\ell_1s_1}(\hat{p}) \sum_{\ell_2s_2} \sum_{m_2s_2} f_{\ell_2s_2m_2s_2}(p) Y_{m_2s_2}^{\ell_2s_2}(\hat{p}),$$

and $Y_{m_1s_1}^{\ell_1s_1}(\hat{p})$ are the usual spherical harmonics. Here and henceforth we will use the notation $p = |p|$, etc., while for four vectors will be written as $p^\mu$. The orbital indices $\ell_1s_1$ and $m_1s_1$ and the radial functions $f_{\ell_1s_1m_1s_1}(p)$ depend on the spin variables $s_1$ and $s_2$. In the rest frame, the operators $\hat{L}_{1z}(q)$ and $\hat{L}_{2z}(q)$ can be expressed in terms of the orbital angular momentum operator, $\hat{L}(q)$, of the relative motion,

$$\hat{L}_{1z}(q) = \frac{m_2}{m_1 + m_2} \hat{L}_z(q), \quad \hat{L}_{2z}(q) = \frac{m_1}{m_1 + m_2} \hat{L}_z(q).$$

The substitution of the partial-wave expansion (20) into the rest-frame form of ansatz (6) leads to two categories of relations among the adjustable functions $F_{s_1s_2}(p)$.

(i) The spin-mixed (quasisinglet and quasitriplet) states. In this case we have $\ell_1s_1 = \ell = J$, and the general solution under the condition of well-defined $\hat{P}$, $\hat{J}$, $\hat{J}_z$, and $\hat{P}$ can be expressed with the help of Dirac $\Gamma$ matrices as [18]

$$F_{s_1s_2}(p) = \bar{u}_{p \ell_1} \Gamma^{(\ell_1)}_{\ell_1s_1m_1s_1}(\hat{P}) V_{-\ell_2s_2} f_{-\ell_2s_2}(p) + \bar{u}_{p \ell_1} \Gamma^{(\ell_1)}_{\ell_1s_1m_1s_1}(\hat{P}) V_{-\ell_2s_2} f_{-\ell_2s_2}(p).$$

(ii) The $\ell$-mixed triplet states. These states occur for $\ell_1s_1 = \ell = J \mp 1$. Their radial decomposition can be written as

$$F_{s_1s_2}(p) = \bar{u}_{p \ell_1} \Gamma^{(\ell_1)}_{\ell_1s_1m_1s_1}(\hat{P}) V_{-\ell_2s_2} f_{-\ell_2s_2}(p) + \bar{u}_{p \ell_1} \Gamma^{(\ell_1)}_{\ell_1s_1m_1s_1}(\hat{P}) V_{-\ell_2s_2} f_{-\ell_2s_2}(p).$$

The system in these states is characterized by $J$, $m_J$, and $P = (-1)^J$, and $\ell$ is not a good quantum number. The two radial functions $f_{\ell_1s_1}(p)$ and $f_{\ell_2s_2}(p)$ correspond to the cases $\ell = J - 1$ and $\ell = J + 1$. Mixing of this type occurs only for principal quantum number $n \geq 3$.

From the variational method we obtain a system of coupled radial equations expressed in matrix form as

$$\left( \omega_p + \Omega_p - E \right) f(p) = \frac{m_1m_2}{(2\pi)^3} \int \frac{q^2 dq}{\sqrt{\omega_p \omega_q \Omega_p \Omega_q}} K(p,q) f(q),$$

where $\omega_p = p^2 + m_1^2$ and $\Omega_p = p^2 + m_2^2$, and $q = |q|$ as already mentioned. Here $f(p)$ and $K(p,q)$ are matrices composed of radial functions and kernels, respectively. The kernel matrix $K = K_{\text{opt}} + K_{\text{ext}}$ is made up of one-photon-exchange and external-field parts. Explicit expressions for $K_{\text{opt}}$ can be found in Ref. [18], while the external-field contributions are calculated in this work.

For the spin-mixed states the two-component Fock-space amplitude is given as

$$F(p) = \begin{bmatrix} f_j^{(e)}(p) \\ f_j^{(tr)}(p) \end{bmatrix}.$$

The equations imply a mixing of spin and radial variables, and the radial equations are usually coupled. We apply a unitary transformation with rotation angle $\beta$ to the spin part of function (22) to diagonalize the kernel matrix. The diagonalization can be carried out for arbitrary $p$ and $q$ [cf. Eq. (A7) in the Appendix], and defines a new quasispin basis,

$$|s_1, s_2, \ell, \bar{s}, J, m_J \rangle = C_1 |s_1, s_2, \ell, S = 0, J, m_J \rangle + C_2 |s_1, s_2, \ell, S = 1, J, m_J \rangle,$$

where $\ell = J, S$ is the total spin of the system, and $\bar{s} = 0$ for quasisinglet and $\bar{s} = 1$ for quasitriplet states. The coefficients used to express the new basis states in terms of the previously defined singlet and triplet states are found to be $C_1 = \sqrt{(1 + \xi)/2}$, $C_2 = -\sqrt{(1 - \xi)/2}$, for the quasi-singlet states, and $C_1 = \sqrt{(1 - \xi)/2}$, $C_2 = \sqrt{(1 + \xi)/2}$ for the quasitriplet states. Here the rotation angle $\beta$ has been replaced for convenience according to $\tan 2\beta = \sqrt{1 - \xi^2}/\xi$.

The quasisinglet and quasitriplet states are both characterized by the same quantum numbers $J$, $m_J$, and $P = (-1)^J$, and they mix the states given in the LS coupling representation. The states are labeled for convenience not by the quasispin $z$ projection $t_{\bar{s}} = \mp 1/2$, but rather by $\bar{s} = t_{\bar{s}} + 1/2$, which takes on the values of 0, 1. In the Appendix the kernels for spin-mixed states are given explicitly in order to solve for the angle $\beta$, i.e., to determine the $\xi$ values.

In the limit $m_2 \gg m_1$ the total angular momenta of the first and the second particles are $j_1 = \ell_1 \pm 1/2$, $j_2 = s_2 = 1/2$, where $\ell_1 = \ell$. In this case $j_1$ can be used as a good quantum number, and the role of the indices $\bar{s}, \bar{s}$ are played by $j_1 = \ell_1 \pm 1/2$ and $j_1 = \ell_1 - 1/2$, respectively. In this case the coefficients $C_1$ and $C_2$ reduce to $C-G$ coefficients,

$$C_{1,2} = (-1)^{1/2 + 1/2 + \ell_1 s_1} \sqrt{(2S + 1)(2j_1 + 1)} \begin{bmatrix} 1/2 & 1/2 & S \\ \ell_1 & \ell_1 & j_1 \end{bmatrix}.$$
Note that the one-body limit corresponds to the $j_1 j_2$ coupling representation, which cannot be used in the general case of arbitrary masses since $j_1$ and $j_2$ are not independent (they are related through the common angular momentum $\ell$). For positronium the quasistates become true singlet ($C_2=0$) and triplet ($C_1=0$) states with different charge conjugation quantum numbers.

We now proceed to calculate the kernels $K_{mn}^{ext}(p,q)$ associated with the classical external field $A_\mu^{ext}$. Using Eq. (9) for $M_{\epsilon_1 \epsilon_2 \sigma_1 \sigma_2}^{ext}$ taken in the rest frame, we obtain

$$K_{mn}^{ext}(p,q) = -\frac{(\pi/2)^{3/2}}{N(m_1 m_2)} \int d^3\hat{p} d^3\hat{q} \times \text{Tr} \left( \frac{Q_1}{\sqrt{\Omega}} A_\mu^{ext}(p-q) (\gamma^\mu q_\lambda + m_1) \Gamma^m(\hat{q}) (\gamma^\mu q_\lambda - m_2) \Gamma^m(\hat{p}) - \frac{Q_2}{\sqrt{\Omega}} A_\mu^{ext}(p-q) (\gamma^\mu q_\lambda + m_1) \Gamma^m(\hat{q}) (\gamma^\mu q_\lambda - m_2) \Gamma^m(\hat{p}) \right),$$

(28)

where $N$-normalization factor, $q=(\omega_p,q)$, and $\hat{q}=(\Omega_q,-q)$. The $\Gamma$ matrices correspond to the various $J^P$ states. The evaluation of these kernels would allow one to obtain all relativistic corrections to the $g$ factor (5); however, this is a formidable task. To determine the lowest-order effect it is sufficient to use the nonrelativistic limit ($q^2/m^2 \ll 1$). In this case the kernels (28) take the form

$$K_{mn}^{ext}(p,q) = -\frac{(\pi/2)^{3/2}}{N} \int d^3\hat{p} d^3\hat{q} \times \text{Tr} \left( \frac{Q_1 A_\mu^{ext}(p-q) (\gamma^\mu + I) \gamma^\mu (\gamma^\mu + I) \Gamma^m(\hat{q}) (\gamma^\mu - I) \Gamma^m(\hat{p}) - \frac{Q_2 A_\mu^{ext}(p-q) (\gamma^\mu + I) \gamma^\mu (\gamma^\mu - I) \Gamma^m(\hat{q}) (\gamma^\mu - I) \Gamma^m(\hat{p}) }{2} \right).$$

(29)

These are evaluated for a stationary uniform magnetic field (10). The results are given separately for the following two types of states.

(i) The spin-mixed states [$\ell=J,J\pm 1, \mathcal{P}=(-1)^{J+1}$]. In contrast to $K^{(osp)}(p,q)$ the kernel matrix $K^{(ext)}(p,q)$ is not diagonal in the basis of the quasisinglet $|s g_q \rangle$ and quasitriplet $|t r_q \rangle$ states, and can be written as

$$K_{11}^{(ext)}(p,q) = -\frac{(2\pi)^3}{2c} \left( \begin{array}{c} Q_1 \left[ \frac{1}{\sqrt{J+1}} \right] \frac{m_2}{m_1 + m_2} + \frac{g_{s_1}}{2} \left( \frac{1}{2J(J+1)} \right) - \frac{m_1 - m_2}{m_1 + m_2} \frac{\xi}{2} \right) B_{m_1},$$

$$K_{22}^{(ext)}(p,q) = -\frac{(2\pi)^3}{2c} \left( \begin{array}{c} Q_2 \left[ \frac{1}{\sqrt{J+1}} \right] \frac{m_2}{m_1 + m_2} + \frac{g_{s_2}}{2} \left( \frac{1}{2J(J+1)} \right) + \frac{m_1 - m_2}{m_1 + m_2} \frac{\xi}{2} \right) B_{m_2},$$

$$K_{12}^{(ext)}(p,q) = K_{21}^{(ext)}(p,q) = -\frac{(2\pi)^3}{2c} \left( \begin{array}{c} Q_1 \left[ \frac{1}{\sqrt{J+1}} \right] \frac{m_2}{m_1 + m_2} + \frac{g_{s_1}}{2} \left( \frac{1}{2J(J+1)} \right) - \frac{m_1 - m_2}{m_1 + m_2} \frac{\xi}{2} \right) B_{m_1},$$

$$K_{21}^{(ext)}(p,q) = K_{12}^{(ext)}(p,q) = -\frac{(2\pi)^3}{2c} \left( \begin{array}{c} Q_2 \left[ \frac{1}{\sqrt{J+1}} \right] \frac{m_2}{m_1 + m_2} + \frac{g_{s_2}}{2} \left( \frac{1}{2J(J+1)} \right) + \frac{m_1 - m_2}{m_1 + m_2} \frac{\xi}{2} \right) B_{m_2}. $$

(30-32)

Thus, it couples the system (24).

(ii) The pure triplet and $\ell$-mixed states [$\ell=J\pm 1, \mathcal{P}=(-1)^J$]. The system (24) cannot be decoupled for these states, and the matrix $K^{(osp)}(p,q)$ is not diagonal [19]. The magnetic part of the kernel is, however, diagonal,

$$K^{(ext)}(p,q) = -\frac{(2\pi)^3}{2c} \left( \begin{array}{c} Q_1 \left[ \frac{1}{m_1} - \frac{Q_2}{m_2} \right] \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right] B_{m_1}. $$

(33)

All kernels $K^{(ext)}$ vanish in the case of equal masses and opposite charges ($Q_1=Q_2$), as occurs in the positronium case, where magnetic effects appear only in $O(B^2)$ [23].
IV. HFS TO $O(\alpha^4)$ ORDER IN A MAGNETIC FIELD

To obtain results for energy levels to $O(\alpha^4)$ we solve the radial equations (24) perturbatively using hydrogenlike radial functions [nonrelativistic Schrödinger form $f_{nJm_j}(p)$] in momentum space [9]. The energy eigenvalues can be calculated from the matrix equation, which follows from Eq. (24)

$$E \int p^2 dp F^*(p)F(p) = \int p^2 dp (\omega_p + \Omega_p) F^*(p)F(p) + \frac{m_1m_2}{(2\pi)^3} \int_0^\infty \frac{p^2 dp}{\sqrt{\omega_p \Omega_p}} \int_0^\infty \frac{q^2 dq}{\sqrt{\omega_q \Omega_q}} F^*(p)K(p,q)F(q).$$

(34)

If the system (24) has been decoupled, or the contribution of nondiagonal elements of the $K(p,q)$ matrix with given radial functions in (34) is zero, Eq. (34) immediately gives the perturbative solution for the energy levels. As shown in Ref. [19], the contribution of the nondiagonal elements $K^{\text{ext}}_{12}$ and $K^{\text{ext}}_{21}$ in Eq. (34) to order $O(\alpha^4)$ is zero for the $\ell$-mixing states. Thus, in the present scheme the energy corrections for the $\ell$-mixing states can be calculated independently for $\ell=J-1$ and $\ell=J+1$ states. As a result, all triplet states with $\ell=J-1$ can be treated as pure states. In the case of spin-mixed states the kernel matrix $K^{\text{ext}}$ has been diagonalized in the basis of quasistates (26), however, the magnetic part of the interaction gives rise to the nondiagonal terms (32).

Since we are solving the system (34) perturbatively, we can use a new basis $|\text{ext}\rangle = C_1^*|sg\rangle + C_2^*|tr\rangle$, which mixes the quasistates with arbitrary constants $C_1$ and $C_2$. This leads to a two-level problem with the solution $E_{n,J,m_j} = (H_{11} + H_{22})/2 \pm \sqrt{((H_{11} - H_{22})/2)^2 + H_{12}H_{21}/4}$, where $H_{11} = H_{0,11}^{\text{ext}} + H_{0,11}^{\text{int}}$, $H_{22} = H_{0,22}^{\text{ext}} + H_{0,22}^{\text{int}}$, and $H_{12} = H_{0,12}^{\text{ext}} + H_{0,12}^{\text{int}}$. In our case, $|H_{11} - H_{22}| \ll H_{12}H_{21}$, because the difference $|H_{11} - H_{22}|$ is of the order of the fine structure, which dominates over the hyperfine splitting and the magnetic perturbation $H_{12}$. Therefore, we can approximate $E_{n,J,m_j} \approx H_{11}, H_{22}$.

The results are presented in the form

$$\Delta E_{n,J,m_j} = E_{n,J,m_j} - (m_1 + m_2) + \frac{(Z\alpha)^2 m_j}{2n^2} = \Delta E_{n,J}(\alpha^4) + \Delta E_{n,J,m_j}^{\text{ext}},$$

(35)

where $Q_2 = ZQ$. The energy corrections $\Delta E_{n,J}(\alpha^4)$ due to the kernels $K^{\text{ext}}(p,q)$ were obtained previously [19]. The corrections $\Delta E_{n,J,m_j}^{\text{ext}}(\alpha^4)$ contain spin-spin interactions that lead to the HFS, which is illustrated in Fig. 1 for the low-lying excited states. A detailed analysis of the HFS to $O(\alpha^4)$ is provided in Ref. [19]. We note that the HFS of the 1S$_{1/2}$ and 2S$_{1/2}$ states is obtained in agreement with the known Fermi splittings [9], i.e., $\Delta E_{\text{HFS}}(1S_{1/2}) = (Z\alpha)^4 m_j (8m_j/3M)$, and $\Delta E_{\text{HFS}}(2S_{1/2}) = (Z\alpha)^4 m_j (m_j/3M)$, where $M = m_1 + m_2$. The HFS of states with $\ell > 0$, however, is more complicated [19]. In standard spectroscopic notation it has the form

where the quantity $\xi$ is defined by Eq. (A8), but with the quantum number $J$ replaced by $\ell$. The formulas (36) and (37) are valid for all quantum numbers $n$, $\ell$ and for any mass values $m_1,m_2$. The weak external field further splits the energy levels. Equations (36) and (37) give excellent agreement with experiment for the HFS [19].

The energy corrections $\Delta E_{J,J,m_j}^{\text{ext}}$ remove the degeneracy with respect to the $m_j$ quantum number. The solution of Eq. (34) in the above-made approximation can be written in the form of Eq. (1) for all states.
For all pure states ($\ell=J \pm 1$) we obtain the following results:

for $\ell=J-1$:

$$g_{1,2} = 1 - \frac{m_{1,2}}{m_1 + m_2} \frac{J - 1}{J} + \left( \frac{g_{s_{1,2}}}{2} - 1 \right) \frac{1}{J},$$

(38)

for $\ell=J+1$:

$$g_{1,2} = 1 - \frac{m_{1,2}}{m_1 + m_2} \frac{J + 2}{J + 1} + \left( \frac{g_{s_{1,2}}}{2} - 1 \right) \frac{1}{J + 1}.$$  

(39)

For spin–mixed states $\ell\neq J \neq 0$ the solution of Eq. (34), as mentioned, reduces to a standard two-energy level problem. The diagonal elements of the matrix kernel give the first-order Zeeman splitting [in $O(B)$] in the quassispin representation (26), which was used to derive the HFS energies (36) and (37). Note that the nondiagonal elements give a contribution to higher-order Zeeman splitting corrections.

To first order in the magnetic field strength we obtain the Landé factors to be

$$g_1 = \frac{m_2}{m_1 + m_2} \left( 1 - \frac{1 \pm \xi}{2J(J+1)} \right) + \frac{g_{2s}}{2} \left( \frac{1 \pm \xi}{2J(J+1)} \right) \pm \frac{1}{2} \frac{m_1 - m_2}{m_1 + m_2} \xi,$$

(40)

$$g_2 = \frac{m_1}{m_1 + m_2} \left( 1 - \frac{1 \mp \xi}{2J(J+1)} \right) + \frac{g_{2s}}{2} \left( \frac{1 \mp \xi}{2J(J+1)} \right) \pm \frac{1}{2} \frac{m_1 - m_2}{m_1 + m_2} \xi,$$

(41)

where the upper sign is taken for $sg_{q}$ and the lower sign for $tr_{q}$ states, respectively. Our expressions (40) and (41) are symmetrical with respect to the masses of the two particles. Obviously all these first-order Zeeman corrections $\Delta E_{J,m,J,m}^{ext}$ vanish for the positronium case ($m_1=m_2=m_p$, $Z=1$), as expected. The intrinsic factors $g_{s_{1,2}}$ associated with the spins of the individual particles can include QED corrections.

In the case when $m_2 \gg m_1$ our general results agree with the result from Eqs. (2) and (4) in which the orbital motion of the heavy particle is ignored. It is only in this limit [as discussed below Eq. (27)], that the total angular momenta of the individual particles are not related through the common angular momentum $\ell$, and can be written as $j_1=\ell \pm 1/2$, and $j_2=1/2$. In $j_1\cdot j_2$ coupling, the eigenstates are taken to be the eigenstates of the operators $\hat{J}_z = (\hat{L}_z + \hat{s}_z)/2$, $\hat{J}_x = \hat{L}_x + \hat{s}_x$, $\hat{J}_y$, and $\hat{J}_z$, and are designated as $|j_1j_2Jm,J\rangle$ in contrast to the spin-mixed $|LSJm\rangle$ and pure states $|LSJM\rangle$, which diagonalize the expectation value of the Hamiltonian to order $O(\alpha^2)$. To facilitate the comparison we make the following replacement of quantum numbers: $F \rightarrow J$, $J \rightarrow j_1$, $L \rightarrow \ell_1 = \ell$, $s \rightarrow s_1$, $l \rightarrow s_2$. It follows that for all pure states $\ell = J \mp 1$, formulas (38), (39), and (2)–(4) give the same result, namely,

$$g_1 = 1 + \left( \frac{g_{s_1}}{2} - 1 \right) \frac{1}{J}, \quad g_2 = \frac{g_{s_1}}{2} \frac{1}{J},$$

(42)

for $\ell = j_1 - 1/2 = J - 1$ and

$$g_1 = 1 - \left( \frac{g_{s_1}}{2} - 1 \right) \frac{1}{J + 1}, \quad g_2 = - \frac{g_{s_1}}{2} \frac{1}{J + 1},$$

(43)

for $\ell = j_1 + 1/2 = J + 1$.

In the limit $m_2 \gg m_1$ the energy levels of spin-mixed states $\Delta E_{J,m,J,m}^{ext}$ reduce to $\Delta E_{J,m,J,m}^{ext}$ and $\Delta E_{J,m,J,m}^{ext}$, respectively, and the Landé factors given by Eqs. (40) and (41) take the form

$$g_1 = \frac{2J + 3}{2J + 1} + \left( \frac{g_{s_1}}{2} - 1 \right) \frac{1}{J}, \quad g_2 = - \frac{1}{J + 1} - \left( \frac{g_{s_1}}{2} - 1 \right) \frac{1}{J + 1},$$

(44)

for $\ell = j_1 + 1/2 = J(tr_{q})$, and

$$g_1 = \frac{2J - 1}{2J + 1} + \left( \frac{g_{s_1}}{2} - 1 \right) \frac{1}{J + 1}, \quad g_2 = \frac{1}{J} + \left( \frac{g_{s_1}}{2} - 1 \right) \frac{1}{J}$$

(45)

for $(sg_q)$ states, and

$$g_1 = \frac{2J - 1}{2J + 1} - \left( \frac{g_{s_1}}{2} - 1 \right) \frac{2J - 1}{(2J + 1)(J + 1)}$$

(46)

for $(tr_{q})$ states. This result agrees with Eqs. (44) and (45) only in the particular case of $g_{s_1} = 2$. Note that most theoretical and experimental results are concerned with $nS_{1/2}(J = 1)$ states for which the “mass ratio” correction in Eq. (38) disappears. Thus our results will be most useful for $\ell > 0$ states.

In Tables I–III we present results of our calculations of the g factors for the first excited states in hydrogen, muonium, and muonic hydrogen respectively. Only states with nonzero total angular momentum are included. Equation (40) and (41) are used for the spin-mixed states $P_{1/2,J_m}(1)$, $P_{3/2,J_m}(1)$, $P_{1/2,J_m}(2)$, $P_{3/2,J_m}(2)$, $D_{3/2,J_m}(2)$, $D_{5/2,J_m}(2)$. Equation (38) is used for the pure state $P_{3/2,J_m}(2)$.

Our calculations (given to five digits after the decimal point) are to be compared with the $(m_2 \rightarrow \infty)$ results (2) and (4). Upper values for each $g$ factor have taken into account the following anomalous magnetic moment values: $g_{e/2} = 1.00118$, $g_{p/2} = 1.792847$, and $g_{d/2} = 1.001166$, [1, 9, 14]. The intrinsic proton anomaly reflects the fact that it is not a
respectively. Results from the present calculation, Eqs. (38) and (40) for electrons, are compared with Eq. (2) in the top half of the table. For protons the bottom half displays the present results from Eqs. (38) and (41) in comparison with Eq. (4). Each row contains in the upper part the Landé factor where the intrinsic $g_s$ value is corrected for the anomaly (see text), while the numbers below are based upon the Dirac value $g_s = 2$.

<table>
<thead>
<tr>
<th>$pe^-$</th>
<th>$P_{1/2}(J=1)$</th>
<th>$P_{3/2}(J=1)$</th>
<th>$P_{3/2}(J=2)$</th>
<th>$D_{5/2}(J=2)$</th>
<th>$D_{3/2}(J=2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_1$ using Eqs. (38) and (40)</td>
<td>0.33237</td>
<td>1.66740</td>
<td>1.00032</td>
<td>0.59912</td>
<td>1.40008</td>
</tr>
<tr>
<td>$g_1$ using Eq. (2)</td>
<td>0.33296</td>
<td>1.66622</td>
<td>0.99973</td>
<td>0.59951</td>
<td>1.39949</td>
</tr>
<tr>
<td>$g_2$ using Eqs. (38) and (41)</td>
<td>1.79321</td>
<td>-0.89597</td>
<td>0.89670</td>
<td>0.89691</td>
<td>-0.59711</td>
</tr>
<tr>
<td>$g_2$ using Eq. (4)</td>
<td>1.00036</td>
<td>-0.49955</td>
<td>0.50027</td>
<td>0.50049</td>
<td>-0.33283</td>
</tr>
</tbody>
</table>

As mentioned above, our results are applicable only in low magnetic fields, such that the hyperfine energy splitting exceeds the Zeeman splitting, namely,

$$B \ll \frac{\Delta_{E_{HFS}}(n, \ell)}{\mu_B g}. \quad (48)$$

Thus, formula (48), for $2P_{3/2}$ states, requires that $B \ll 300 \text{ G}$ for muonium and $B \ll 100 \text{ G}$ for hydrogen.

V. CONCLUSION

We have used the Hamiltonian variational method in reformulated QED to derive relativistic stationary-state equations for two-fermion systems in an external magnetic field. These equations can include interactions to any order of the coupling constant, at least in principle. The classification of the states follows naturally from the conserved quantum numbers that appear in the trial state (6). For given total angular momentum $J$ there are, in general, coupled equations, both for mixed-spin states, and for triplet mixed-$\ell$ states [cf. Eq. (24)]. We present explicit forms for the kernels (momentum-space potentials) for the case of a constant, weak external magnetic field.

We solved the radial equations perturbatively to obtain the Zeeman splitting of the HFS to order $O(a_0^4)$, and calculated the $g$ factors for the system of two bound fermions. Our

<table>
<thead>
<tr>
<th>$\mu^+e^-$</th>
<th>$P_{1/2}(J=1)$</th>
<th>$P_{3/2}(J=1)$</th>
<th>$P_{3/2}(J=2)$</th>
<th>$D_{5/2}(J=2)$</th>
<th>$D_{3/2}(J=2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_1$ using Eqs. (38) and (40)</td>
<td>0.329451</td>
<td>1.66392</td>
<td>0.99818</td>
<td>0.59527</td>
<td>1.39610</td>
</tr>
<tr>
<td>$g_1$ using Eq. (2)</td>
<td>0.33004</td>
<td>1.66274</td>
<td>0.99759</td>
<td>0.59566</td>
<td>1.39551</td>
</tr>
<tr>
<td>$g_2$ using Eqs. (38)–(41)</td>
<td>1.00344</td>
<td>-0.49657</td>
<td>0.50299</td>
<td>0.50491</td>
<td>-0.32923</td>
</tr>
<tr>
<td>$g_2$ using Eq. (4)</td>
<td>1.00117</td>
<td>-0.50058</td>
<td>0.50058</td>
<td>0.50058</td>
<td>-0.33372</td>
</tr>
</tbody>
</table>

For atomic hydrogen the effect is smallest due to the small $\ell/p$ mass ratio. Given that atomic spectroscopy is far more advanced in hydrogen than in muonic atoms one should not neglect these corrections. For the two above-mentioned states that are most affected we observe about 0.1% deviations in the electron and proton Landé factors, respectively.
results are applicable to all states (i.e., for all quantum numbers) and any fermion masses. In the limit \( m_2 \gg m_1 \) our formulas reproduce the well-known g-factor result. For the spin-mixed states, however, Eq. (2) is found to be not exact if the intrinsic magnetic moment is different from the Dirac value \( g_1 = 2 \).

**ACKNOWLEDGMENT**

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**APPENDIX: ONE-PHOTON EXCHANGE KERNELS FOR THE SPIN-MIXED STATES TO ORDER \( \alpha^4 \)**

We use the notation \( z = (p^2 + q^2)/2pq \), and \( Q_j(z) \) is the Legendre function of the second kind [24]. The contributions of the various terms to the kernel are as follows [\( \ell = J(J+1), P = (-1)^{J+1} \)]:

(i) orbital term:

\[
K_{11}^{(o)}(p,q) = \frac{2 \pi Q_1 Q_2}{pq} Q_j(z) + \frac{\pi Q_1 Q_2}{2m_1 m_2} \left( \frac{m_1 + m_2}{m_2 + m_1} - (J - 1) \right) \times \left( \frac{p + q}{p} \right) Q_j(z) + 2(J+1)Q_{J+1}(z),
\]

(ii) spin-orbit interaction:

\[
K_{12}^{(o)}(p,q) = 0,
\]

\[
K_{12}^{(s-o)}(p,q) = - \frac{\pi Q_1 Q_2}{2m_1 m_2} \left\{ \frac{m_1 - m_2}{m_2 - m_1} \right\} \frac{2J+1}{2J+1} \left[ Q_{J+1}(z) - Q_{J-1}(z) \right],
\]

\[
K_{22}^{(s-o)}(p,q) = - \frac{\pi Q_1 Q_2}{2m_1 m_2} \left\{ \frac{m_1 + m_2}{m_2 + m_1} + 4 \right\} \frac{1}{2J+1} \left[ Q_{J+1}(z) - Q_{J-1}(z) \right].
\]

(iii) spin-spin interaction:

\[
K_{11}^{(s-s)}(p,q) = \frac{\pi Q_1 Q_2}{m_1 m_2} \frac{1}{2J+1} \left[ Q_{J+1}(z) - Q_{J-1}(z) \right].
\]

The diagonalization condition

\[
\tan 2\beta [K_{22}(p,q) - K_{11}(p,q)] = 2K_{12}(p,q)
\]

determines the parameters \( \beta \) and \( \xi \):

\[
\tan 2\beta = 2 \left| \frac{m_1 - m_2}{m_1 + m_2} \right| \frac{\sqrt{J(J+1)}}{\xi},
\]

\[
\xi = \left[ 4 \left\{ \frac{m_1 - m_2}{m_1 + m_2} \right\}^2 J(J+1) + 1 \right]^{-1/2}.
\]

Therefore, we obtain the diagonalized kernels for the quasisstates

\[
K^{(s-o)}(p,q) = K_{11}^{(o)} + \frac{\xi + 1}{\xi - 1} K_{12}^{(s-o)}
\]

\[
= \frac{2 \pi Q_1 Q_2}{pq} Q_j(z) + \frac{\pi Q_1 Q_2}{2m_1 m_2} \left( \frac{m_1 + m_2}{m_2 + m_1} - (J - 1) \right) \times \left( \frac{p + q}{p} \right) Q_j(z) + 2(J+1)Q_{J+1}(z)
\]

\[
- \frac{\pi Q_1 Q_2}{2m_1 m_2} \frac{\xi + 1}{\xi (2J+1)} \left[ Q_{J+1}(z) - Q_{J-1}(z) \right].
\]


