Circular motion and the unit circle

Motion on a circle: 2 dimensions \( \vec{r}(t) = x(t) \hat{i} + y(t) \hat{j} \)

constrained by: \( x(t)^2 + y(t)^2 = R^2 \) \( \leftarrow \) radius

should be described using a single degree of freedom = one real variable \( \rightarrow \) polar angle \( \theta(t) \)

Unit circle (mathematics)

\( \theta = \text{angle in degrees} \)
\( \theta = \text{arc length in radians} \)

Conversion: \( 180^\circ = \pi \) radians
\( \pi = 3.14(159265...) \)

\[ x = \cos \theta \]
\[ y = \sin \theta \]

As \( \theta \) varies from \( 0 \ldots 2\pi \):

We need \( \sin, \cos \) and \( \tan = \frac{\sin}{\cos} \)
in many contexts:

Example 1: \( \sin (wt) = \sin (2\pi ft) \)

= oscillatory (in time) motion, electrical signal

Example 2: Vector \( \vec{v} = v_x \hat{i} + v_y \hat{j} \) to be expressed

as length \& direction: \( \vec{v} = |\vec{v}| = \sqrt{v_x^2 + v_y^2} \)

Direction: use \( \theta = \text{angle from positive x-axis to vector, in pos. math. sense} \) (= counterclockwise CCW).
Inverse trig functions

\[ \frac{v_y}{v_x} = \tan \theta \]

\[ \tan^{-1} \left( \frac{v_y}{v_x} \right) = \tan^{-1}(\tan \theta) = \theta \]

Note: By default, \( \tan^{-1} \) returns values in the range: \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \) (quadrants I, IV)

\[ \frac{v_y}{v_x} > 0 \quad \Rightarrow \quad \text{I} \quad \frac{v_y}{v_x} < 0 \quad \Rightarrow \quad \text{IV} \]

Suppose the vector points into quadrant I, \( v_x < 0, v_y > 0 \)

\( \tan^{-1} \) receives a neg. argument \( \frac{v_y}{v_x} \), answer: \( \theta \) in IV

How do we fix this? Need to add \( \pi \) to \( \theta \)

Alternative: \( \vec{v} = \sqrt{v_x^2 + v_y^2}, \quad \theta = \cos^{-1} \left( \frac{v_x}{v} \right) \) ?

This works in I, II, since \( \cos^{-1} \) maps the interval \((-1, 1)\) into \((\pi, 0)\), i.e., \( 0 < \theta < \pi \) is the range

So, try \( \theta = \sin^{-1} \left( \frac{v_y}{v} \right) \), always yields \( -\frac{\pi}{2} < \theta < \frac{\pi}{2} \)

Always make a sketch, use \( v_x, v_y \) sign info to figure out which quadrant the tip points to, adjust \( \theta \) by adding \( \pi \).
Aside

Small caveat: suppose \(v_x = 1.5\), \(v_y = -0.01\)

\[ \therefore \theta = \tan^{-1}(-0.01) = \tan^{-1}(1.5 \times 10^{-2}) = -1.5 \times 10^{-2} \]

\[ \rightarrow \theta = -0.086^\circ \quad \text{(2 significant digits)} \]

Express as a positive angle (quadrant IV: \(270^\circ < \theta < 360^\circ\))

\[ 360^\circ - 0.86^\circ = 359.14^\circ \quad \text{are we quoting 5 significant digits here? NOT REALLY!} \]

\[ \therefore \text{Advantage in using negative angles when } |\theta| \text{ is small.} \]

Uniform circular motion on circle of radius \(R\):

\[ \vec{r}(t) = R \left[ \cos \theta(t) \hat{i} + \sin \theta(t) \hat{j} \right] \]

when \(t = 0\) implies \(\theta = 0\), the motion starts on the positive x-axis: \(\vec{r}(0) = R \hat{i}\)

Uniform motion is characterized by a period \(T\) (it takes \(T\) seconds to go once around)

The frequency \(f = \frac{1}{T}\) measures revolutions/time

\[ \theta(t) = 2\pi \frac{t}{T} \]

Why? \(0 < t < T\) is one orbit

\[ 0 < \theta < 2\pi \] is one math. period

\[ = 2\pi ft \]

\[ = \omega t \]

defines \(\omega = 2\pi f = \frac{2\pi}{T}\) "circular" frequency