Spinning up disks

uniform circular motion: $\vec{r}(t) = R\cos\omega t\,\hat{\imath} + R\sin\omega t\,\hat{\jmath}$

or: $r = R$, $\vartheta(t) = \omega t$

or $\vartheta(t) = \vartheta_0 + \omega t$

use the analogy to linear motion:

$x(t) = x_0 + v t$

since $r = R$ is fixed we have one degree of freedom in this 2d motion

Q: constant-acceleration rotational motion?

$\vartheta(t) = \vartheta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$ $\quad \alpha = \frac{d\omega}{dt}$ angular acc.

$\omega(t) = \omega_0 + \alpha t$

A: yes, this works: $\vec{r}(t) = R\cos\vartheta(t)\,\hat{\imath} + R\sin\vartheta(t)\,\hat{\jmath}$

$\vec{v}(t) = -R\vartheta'\sin\vartheta(t)\,\hat{\imath} + R\vartheta'\cos\vartheta(t)\,\hat{\jmath}$

$\vec{a}(t)$ is no longer centripetal

$\vec{a}(t) = \vec{a}_{cp}(t) + \vec{a}_{\text{spin up/down}}$ why?

$\vec{a}_{cp}$: as before: $-R(\vartheta')^2\cos\vartheta(t)\,\hat{\imath} - R(\theta')^2\sin\vartheta(t)\,\hat{\jmath}$

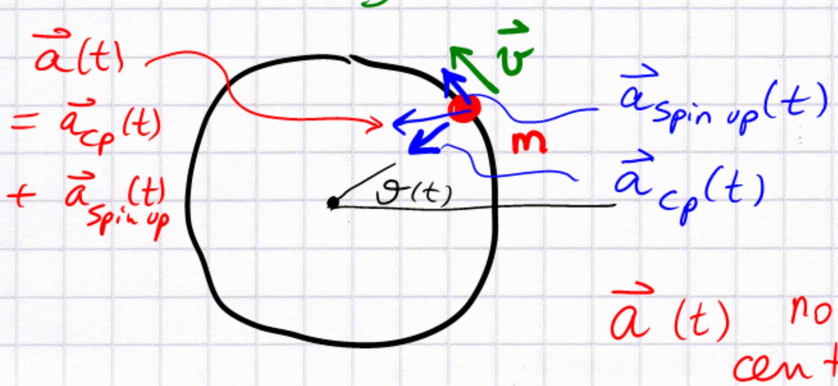
$\vec{a}_{\text{spin up/down}}$: $\to -R\vartheta''\sin\vartheta(t)\hat{\imath} + R\vartheta''\cos\vartheta(t)\,\hat{\jmath}$

used product rule to do:

$\frac{d}{dt}\left(\frac{d\vartheta}{dt}\cdot\sin\vartheta(t)\right)$

$= \frac{d^2\vartheta}{dt^2}\sin\vartheta + \left(\frac{d\vartheta}{dt}\right)^2\cos\vartheta$

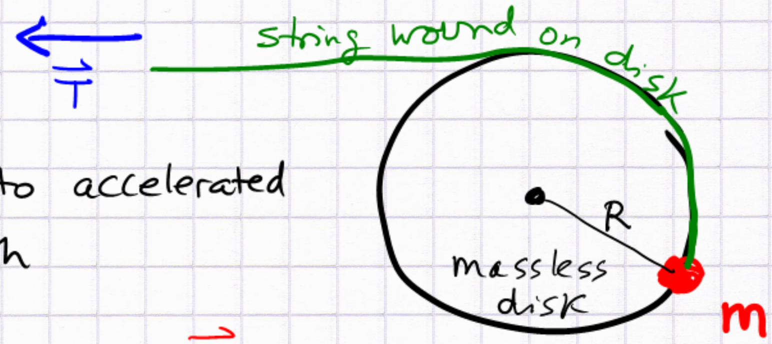
more intuitively:



$\vec{a}(t)$ no longer points to the centre

What could be providing $\vec{a}_{\text{spin up}}$?

Take a light disk, a heavy mass m on the rim, attach a string + pull with tension $\vec{T}$



pulling the string with constant T sets m into accelerated motion on a circular path

- call $\vec{a}_{\text{spin up}} = \vec{a}_{\text{tangential}} = \vec{a}_t$

$a_t \equiv |\vec{a}_t| = \frac{dv}{dt}$ $\quad$ $v =$ speed of m

we want to know: how does the spin rate ω change with time: $\omega'(t) = \alpha$ $\leftarrow$ constant angular accel.

$\omega(t) = \omega_0 + \alpha t$

Newton 2 ? $\boxed{m a_t = T}$ from linear motion (reasonable)

$m \frac{dv}{dt} = T$ $\quad$ but $\omega = \frac{v}{R} \Rightarrow \omega' = \frac{v'}{R}$; $v' = R\omega'$

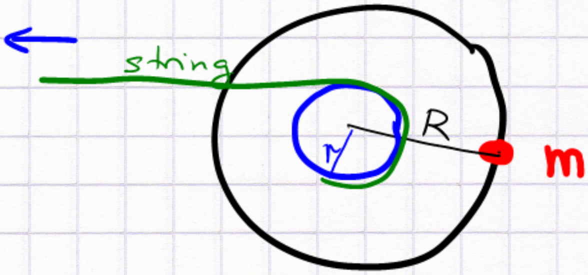
$\therefore$ $\boxed{m R \omega' = T}$ or $\boxed{m R \alpha = T}$

Angular acceleration: $\alpha = \frac{T}{mR}$

Given some tension force T $\rightarrow$ bigger mass $\rightarrow$ less α
$\rightarrow$ bigger R $\rightarrow$ less α

It is harder to spin up a wheel where the mass is further away from the rotation axis!

Now generalize the wheel: m is at radius R,
but wheel → spool (still massless), the string winds
at a different (smaller) radius r



Invoke the Archimedes
lever arm principle:
replace in the previous formula

$$T \to \left(\frac{r}{R}\right) T$$

$r = R$: the same
$r < R$: less effect
$r > R$: easier spin-up

$$\boxed{m R \alpha = \left(\frac{r}{R}\right) T}$$

$$\boxed{\underbrace{m R^2}_{I = mR^2} \; \underbrace{\alpha}_{\text{angular acceleration}} = \underbrace{r T}_{\text{torque (magnitude)}}}$$

Newton-2 for rotation about a fixed axis

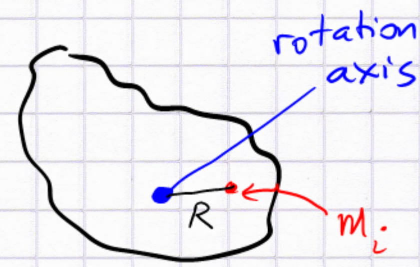
$I = mR^2$
inertia of mass
sitting at radius R

angular acceleration

torque (magnitude); r = arm length at which force applies

Last step: $I = mR^2$ = point mass inertia can
be summed to make up a solid body

$$I = \sum_i m_i R_i^2$$ results in:



Disk: $I = \frac{1}{2} M R^2$
- R: radius of disk
- M: mass " "

Sphere: $I = \frac{2}{5} M R^2$
- M: mass of sphere
- R: radius " "

generally:

$\gamma M R^2$
- R: size scale
- γ: geometric factor