spinning up disks

uniform circular motion: 
\[ \vec{r}(t) = R \cos \omega t \hat{i} + R \sin \omega t \hat{j} \]

or:
\[ r = R \quad \text{or} \quad \vec{r}(t) = \omega t \]

\[ \vec{r}(t) = \vec{r}_0 + \omega t \]

use the analogy to linear motion:
\[ x(t) = x_0 + v_t \]

since \( r = R \) is fixed, we have one degree of freedom in this 2d motion.

Q: Constant-acceleration rotational motion:
\[ \vec{r}(t) = \vec{r}_0 + \omega t + \frac{1}{2} \alpha t^2 \]

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\[ \alpha = \frac{d\omega}{dt} \]

\[ \alpha = \text{angular acc.} \]

\[ o(t) = o_0 + \alpha t \]

A: yes, this works:
\[ \vec{r}(t) = R \cos \theta(t) \hat{i} + R \sin \theta(t) \hat{j} \]

\[ \vec{v}(t) = -R \dot{o} \sin \theta(t) \hat{i} + R \dot{\theta} \cos \theta(t) \hat{j} \]

\[ \vec{a}(t) \text{ is no longer centripetal} \]

\[ \vec{a}(t) = \vec{a}_{cp}(t) + \vec{a}_{spin up} \]

as before:
\[ -R(\dot{\theta})^2 \cos \theta(t) \hat{i} - R(\dot{\theta})^2 \sin \theta(t) \hat{j} \]

more intuitively:
\[ \vec{a}(t) = \vec{a}_{cp}(t) + \vec{a}_{spin up}(t) \]

\[ \vec{a}(t) \text{ no longer points to the centre} \]

\[ \vec{a}(t) = \vec{a}_{cp}(t) + \vec{a}_{spin up}(t) \]
What could be providing $\vec{a}_{\text{spin up}}$?

Take a light disk, a heavy mass $m$ on the rim, attach a string + pull with tension $\vec{T}$

pulling the string with constant $T$ sets $m$ into accelerated motion on a circular path

- call $\vec{a}_{\text{spin up}} = \vec{a}_{\text{tangential}} = \vec{a}_t$

\[ a_t = |\vec{a}_t| = \frac{dv}{dt} \quad v = \text{speed of } m \]

we want to know: how does the spin rate $\omega$ change with time: \[ \omega'(t) = \alpha \] \text{ constant angular accel.}

\[ \omega(t) = \omega_0 + \alpha t \]

Newton 2:

\[ m a_t = T \]

\[ m \frac{dv}{dt} = T \]

but \[ \omega = \frac{v}{R} \]

\[ \Rightarrow \omega' = \frac{v'}{R} ; v' = Rw' \]

\[ \therefore m R \omega' = T \]

or \[ m R \alpha = T \]

Angular acceleration: \[ \alpha = \frac{T}{mR} \]

Given some tension force $T$ \[ \implies \begin{array}{c} \text{bigger mass} \rightarrow \text{less } \alpha \\ \text{bigger } R \rightarrow \text{less } \alpha \end{array} \]

It is harder to spin up a wheel where the mass is further away from the rotation axis!
Now generalize the wheel: \( m \) is at radius \( R \), but wheel \( \rightarrow \) spool (still massless), the string winds at a different (smaller) radius \( r \)

Invoke the Archimedes lever arm principle:
replace in the previous formula
\[
T \rightarrow \left( \frac{r}{R} \right) T
\]

\[
mR^2 \alpha = \left( \frac{r}{R} \right) T
\]

\[
mR^2 \alpha = rT
\]

Newton - 2: for rotation about a fixed axis
\[
\text{torque (magnitude)} \quad \frac{r}{L} \quad \text{angular acceleration at which force applies}
\]

Last step: \( I = mR^2 \) = point mass inertia can be summed to make up a solid body

\[
I = \sum \limits_{i} m_i R_i^2
\]

results in:

\[
\text{Disk: } I = \frac{1}{2} MR^2
\]

\[
\text{Sphere: } I = \frac{2}{5} MR^2
\]

\[
\text{generally: } I = \gamma MR^2
\]

\( \gamma \) size scale geometric factor