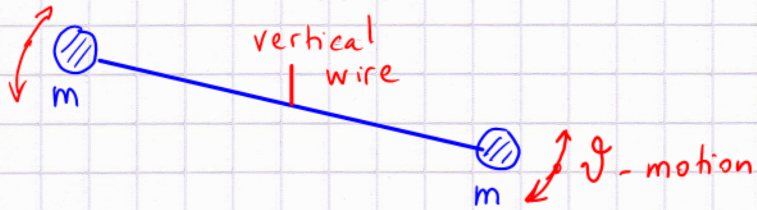
Cavendish torsion balance → measure G

top view:



flat vertical wire → is twisted when masses m rotate by ϑ away from ϑ_0 (equilibrium)

wire provides a restoring torque $-C(\vartheta - \vartheta_0)$ (analogous to Hooke's law for a linear spring)

$\vartheta > \vartheta_0$: it pulls back towards $\vartheta = \vartheta_0$

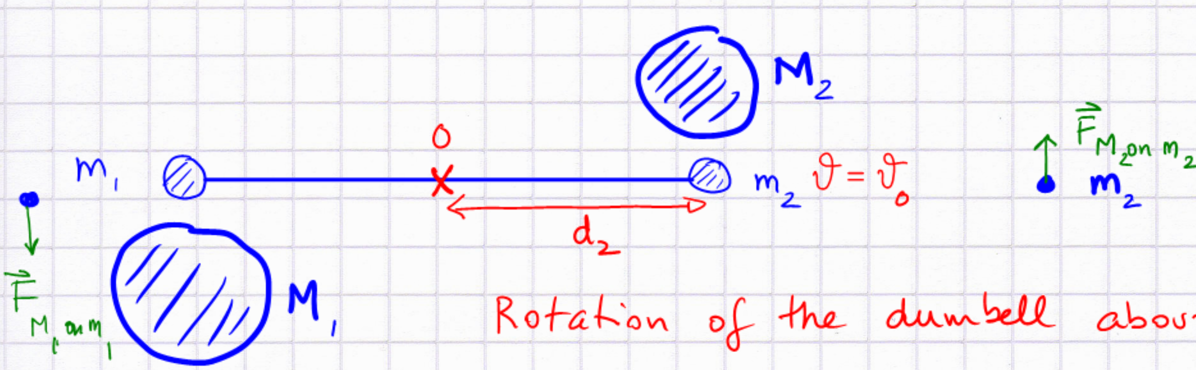
$\vartheta < \vartheta_0$: positive torque (why?) → increases ϑ to bring the wire back to $\vartheta = \vartheta_0$

The result: an oscillator (analogous to slinky or mass attached to spring)

Enclose the wire + rod + masses m in a box
+ attach a mirror to rod → light beam can measure ϑ and amplify signal!

(tiny motions displayed; insensitive to air currents)

Now bring big, heavy (lead) balls close



Rotation of the dumbell about O:

gravitational torque $\tau_2 = d_2 \, F_{M_2 \text{ on } m_2}$ acts to increase ϑ; ϑ grows → $\vartheta > \vartheta_0$ restoring wire twist torque kicks in

Small mass m_2 is accelerated by the stationary M_2

→ at some ϑ ($> \vartheta_0$) value the restoring torque cancels gravitational torque → new equilibrium ϑ_1

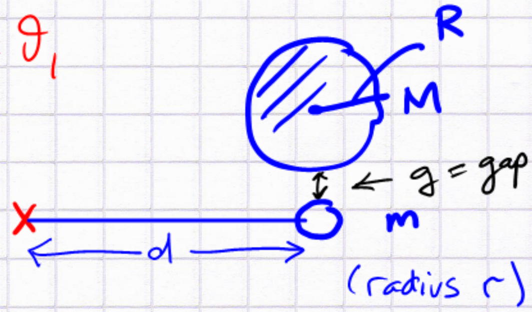
→ when we bring M_2 (and M_1) into position

→ damped oscillatory motion → settles @ ϑ_1

then move the masses to the opposite side + repeat

basic physics: @ equilibrium $\vartheta = \vartheta_1$

$$C(\vartheta_1 - \vartheta_0) = 2d \frac{GMm}{(R+g+r)^2}$$



same contribution from the opposite side

Suppose we measure $\vartheta_1, \vartheta_0, g$ + know C, M, m, R, r

→ we can determine G !

[in practice: C is determined from oscillation period → to be studied in Chapter 11]

$$G \approx 6.7 \times 10^{-11} \ \frac{N\,m^2}{kg^2}$$

Remarkable + tested truth: Newton's law of gravity as measured on cm-distance scale works for $d \lesssim 1\,mm$, in the solar system, in spiral-arm galaxies (? ← dark matter ?), on inter-galactic scales!

Planetary motion

some planets: near circular orbits $\rightarrow$ uniform circular motion

$$m\,\vec{a}_{cp} = -G\frac{mM}{r^2}\hat{r} = -G\frac{mM}{r^3}\vec{r}$$

($\hat{r}$ = unit vector in direction of $\vec{r}$)

$$m\frac{v^2}{r} = G\frac{mM}{r^2}$$

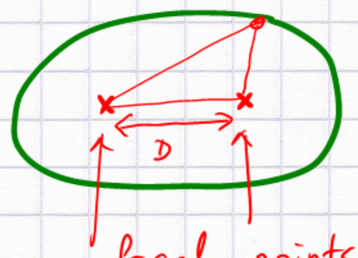
$$v^2 = \frac{GM}{r}$$

$\rightarrow$ earth's speed in motion about sun ($M = M_S$)

$r \approx 1.5 \times 10^{11}$ m

most general bound orbit for $\frac{1}{r^2}$ central force law:

- elliptic orbit



$D \rightarrow 0$: circle

- attracting heavy mass M sits at one of the focal points ×
- acceleration vector always points to that focal point
- speed of m changes on the orbit: $\vec{a}$ plays two roles:
  (a) directional change $\vec{v}(t) \rightarrow \vec{v}(t+\Delta t)$
  (b) $v(t) \neq v(t+\Delta t)$ $\leftarrow$ speed change

m is faster when closer to M and slows when it gets away

How can we model this? (dynamics w/o Calculus)

- discrete time steps $t_k = k\,\Delta t$
- $\vec{r}(t_k) = x(t_k)\,\hat{\imath} + y(t_k)\,\hat{\jmath}$

$$\vec{v}(t_{k+\frac{1}{2}}) = \frac{\vec{r}(t_{k+1}) - \vec{r}(t_k)}{\Delta t}$$ ← average velocity vector

$$\therefore \quad \vec{r}(t_{k+1}) = \vec{r}(t_k) + \Delta t\,\vec{v}(t_{k+1/2}) \quad (1)$$ recursion step

how do we get $\vec{v}(t_{k+\frac{1}{2}})$?

$$\vec{a}(t_k) = \frac{\vec{v}(t_{k+\frac{1}{2}}) - \vec{v}(t_{k-1/2})}{\Delta t} = \frac{\vec{F}_{net}(t_k)}{m}$$

$$\therefore \quad \vec{v}(t_{k+\frac{1}{2}}) = \vec{v}(t_{k-\frac{1}{2}}) + \Delta t\,\frac{\vec{F}_{net}(t_k)}{m}$$

$$\vec{v}(t_{k+\frac{1}{2}}) = \vec{v}(t_{k-\frac{1}{2}}) - \Delta t\,\frac{GM}{r^2(t_k)}\,\hat{r}(t_k) \quad (2)$$ recursion step

Strategy: given two start values:

$k=0$: $\vec{v}(t_{-1/2})$, $\vec{r}(t_0)$

step through the recursion to understand the motion along the ellipse.

Deviations from ellipse: post-Newtonian gravity (Einstein) precession of mercury