Torque and rotational dynamics

Recall the simple discussion of rotational motion about a fixed axis → force and lever arm were assumed to be perpendicular.

Now we generalize.

Vector cross product in \( \mathbb{R}^3 \) (real 3-space)

\( \vec{a}, \vec{b} \) are two vectors (not collinear); the span \( \mathbb{R}^2 \) (the plane)

1) \( \vec{c} = \vec{a} \times \vec{b} \)

is a vector perpendicular to the plane defined by \( \vec{a}, \vec{b} \)

2) \[ |\vec{c}| = |\vec{a}| |\vec{b}| |\sin (\pm \theta, \vec{a}, \vec{b})| = |\vec{a}| |\vec{b}| |\sin \alpha| \]

\( \alpha > 0 \) positive; quadrants I, II
\( \alpha < 0 \) negative; quadrants III, IV

3) We can define the orientation of \( \vec{c} \) by the RH rule

order: \( \vec{a} \) is rotated into \( \vec{b} \) by CCW (math pos.)

operation → thumb, index, middle finger

(\( \vec{a}, \vec{b}, \vec{c} \))

Simplified RH rule:

thumb = \( \vec{c} \) orientation
index ← middle fingers
show \( \vec{a} \rightarrow \vec{b} \)

4) reverse order of \( \vec{a}, \vec{b} \rightarrow \vec{b} \times \vec{a} = -\vec{c} \)

Why? \( \vec{b} \) is rotated into \( \vec{a} \) by CW angle
or by a CCW angle \( > \pi = 180^\circ \)

5) example: \( \hat{i} \times \hat{j} = \hat{k} \)

\( \vec{k} \uparrow \rightarrow \hat{j} \)

RH coordinate system
Define torque
\[ \vec{T} = \vec{r} \times \vec{F} \]

\[ |\vec{T}| = rF \sin \alpha \]
\[ \alpha = \pm \frac{\vec{r} \cdot \vec{F}}{F} \] if \( \vec{r}, \vec{F} \) span \((x,y)\) plane then \( \vec{T} = (0,0,T_2) \)

\( T_2 > 0 \) if \(0 < \alpha < \pi \); \( T_2 < 0 \) if \( \pi < \alpha < 2\pi \)

- \( \vec{T} \) points in the direction of the rotation axis

**Example** \( \frac{1}{2} \) Atwood machine

\[ \vec{r} \times \vec{I} \text{ is out of the plane} \]

\[ T_2 = R M (g-a) \quad \text{Newton}\,2: \quad I_{CM} \alpha = T_2 \]

\[ \frac{1}{2} m R^2 \alpha = R M (g-a) \]

Two unknowns \( (a, \alpha) \)?

Unwinding rope → translation + rotation are locked!

\[ R \Delta \theta = \Delta y : R \alpha \vec{l} = 1gy \]

Eliminate \( a = 1gy \) : \[ \frac{1}{2} m R^2 \alpha = M R \vec{g} - M R \vec{a} \]

\[ (M + \frac{m}{2}) R \alpha = Mg \implies \alpha = \frac{2Mg}{(2M+m)R} \]

\[ a = \frac{2Mg}{2M+m} \]

Linear acceleration of \( M \) is reduced: we can’t turn the disc for free.

Now: increase \( R \rightarrow \alpha \) decreases \( -\frac{1}{R} \) why? Torque increases linearly with \( R \), but inertia increases quadratically.
The angular acceleration \( \alpha \) is interpreted as the 
2- component of \( \ddot{z} = (0, 0, \alpha) \) which causes an 
angular velocity vector to build up: \( \vec{\omega} = (0, 0, \omega) \)

The simplified RH rule reminds us of the orientation of \( \vec{\omega} \) 
(thumb) for a given rotation direction shown by the index, 
middle, and ring fingers

For a CW rotation: \( \tau_2 < 0 \) (into the board)

\[ \vec{\tau} \times \vec{e} \]

\[ \vec{e} = \vec{r} \times \vec{t} \] is into the board

\( \vec{e} \) is into the board

\( \vec{e} \) causes \( \vec{\omega} \) into the board

\[ \vec{e} \sim -\hat{k} \]

\( \tau_2 < 0 \)

\( \omega_2 < 0 \)

This vectorial description

\[ I \ddot{z} = \vec{\tau} \]

allows one to generalize motion from

rotations about a fixed axis to rotations

about an axis that is free to re-orient itself

\( I \) in the eqn will not be a scalar, but a 3-by-3 matrix