Torque and rotational dynamics

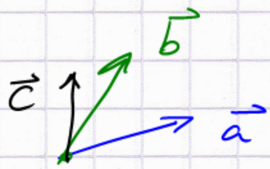
Recall the simple discussion of rotational motion about a fixed axis $\rightarrow$ force and lever arm were assumed to be perpendicular

Now we generalize.

Vector cross product in $\mathbb{R}^3$ (real 3-space)

$\vec{a}, \vec{b}$ are two vectors (not collinear); the span $\mathbb{R}^2$ (the plane)

1) $\vec{c} = \vec{a} \times \vec{b}$



is a vector perpendicular to the plane defined by $\vec{a}, \vec{b}$

2) $|\vec{c}| \equiv |\vec{a}|\,|\vec{b}|\,\underbrace{|\sin(\measuredangle\, \vec{a}, \vec{b})|}_{} = a\,b\,|\sin\alpha|$

$\alpha > 0$ positive; quadrants I, II

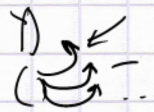
$\alpha < 0$ negative; quadrants III, IV

3) We can define the orientation of $\vec{c}$ by the RH rule

order: $\vec{a}$ is rotated into $\vec{b}$ by CCW (math pos.)

operation $\rightarrow$ thumb, index, middle finger
$(\vec{a}, \vec{b}, \vec{c})$

simplified RH rule:



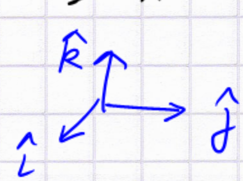
thumb $= \vec{c}$ orientation

index, middle fingers show $\vec{a} \rightarrow \vec{b}$

4) reverse order of $\vec{a}, \vec{b}$ $\rightarrow$ $\vec{b} \times \vec{a} = -\vec{c}$

why? $\vec{b}$ is rotated into $\vec{a}$ by CW angle or by a CCW angle $> \pi = 180°$

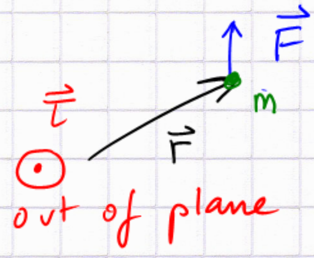
5) example: $\hat{i} \times \hat{j} = \hat{k}$

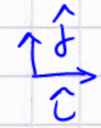


RH coordinate system

Define torque

$\vec{\tau} = \vec{r} \times \vec{F}$





$= (x y)$ plane

$|\vec{\tau}| = r F \sin\alpha$

$\alpha = \sphericalangle\, \vec{r}, \vec{F}$

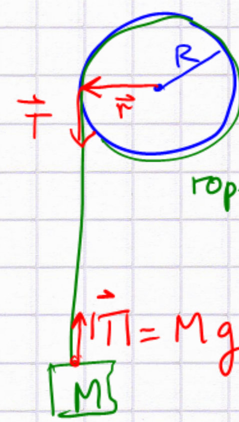
if $\vec{r}, \vec{F}$ span (x,y) plane

then $\vec{\tau} = (0, 0, \tau_z)$

$\tau_z > 0$ if $0 \le \alpha < \pi$; $\tau_z < 0$ if $\pi \le \alpha < 2\pi$

- $\vec{\tau}$ points in the direction of the rotation axis

Example $\frac{1}{2}$ Atwood machine



disk of mass m

$I_{CM} = \frac{1}{2} m R^2$

rope wound

$\vec{r} \times \vec{T}$ is out of the plane ⊙

$\tau_z = R\, M(g-a)$ Newton 2: $I_{CM}\, \alpha = \tau_z$

$|\vec{T}| = Mg - Ma$

why?

$M a_y = F_{net,y}$

$= Mg - T_y$

$\frac{1}{2} m R^2 \alpha = R M (g-a)$

two unknowns (a, α) ?

unwinding rope → translation + rotation are locked! $R\,\Delta\vartheta = \Delta y$ ∴ $R|\alpha| = |a_y|$

eliminate $a = |a_y|$ ∴ $\frac{1}{2} m R^{\cancel{2}} \alpha = M \cancel{R} g - M R^{\cancel{2}} \alpha$

$(M + \frac{m}{2}) R \alpha = M g$ ∴ $\alpha = \frac{2Mg}{(2M+m)R}$; $a = \frac{2Mg}{2M+m}$

Linear acceleration of M is reduced: we can't turn the disc for free

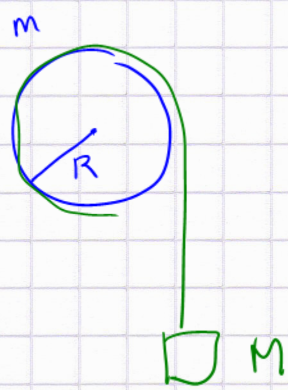
now: increase R → α decreases $\sim \frac{1}{R}$. why? torque increases linearly with R, but inertia increases quadratically

The angular acceleration α is interpreted as the z-component of $\vec{\alpha} = (0, 0, \alpha)$ which causes an angular velocity vector to build up: $\vec{\omega} = (0, 0, \omega)$

The simplified RH rule reminds us of the orientation of $\vec{\omega}$ (thumb) for a given rotation direction shown by the index, middle, and ring fingers

For a CW rotation: $\tau_z < 0$ (into the board)

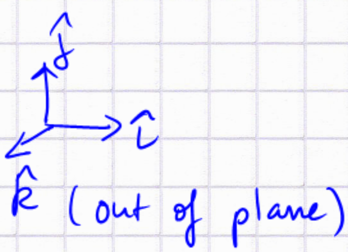
$\otimes \ \vec{\tau}$



$\vec{\tau} = \vec{r} \times \vec{T}$ is into the board

$\rightarrow$ $\vec{\alpha}$ is into the board

$\rightarrow$ causes $\vec{\omega}$ into the board



$\vec{\tau} \sim -\hat{k}$

$\tau_z < 0$ $\quad\therefore\ \alpha_z < 0$

$\therefore\ \omega_z < 0$

$\vec{\omega}$ $\otimes$ into the plane

This vectorial description

$I \vec{\alpha} = \vec{\tau}$ allows one to generalize motion from rotations about a fixed axis to rotations about an axis that is free to re-orient itself

I in the eqn will not be a scalar, but a 3-by-3 matrix