Magnetic field: energy content

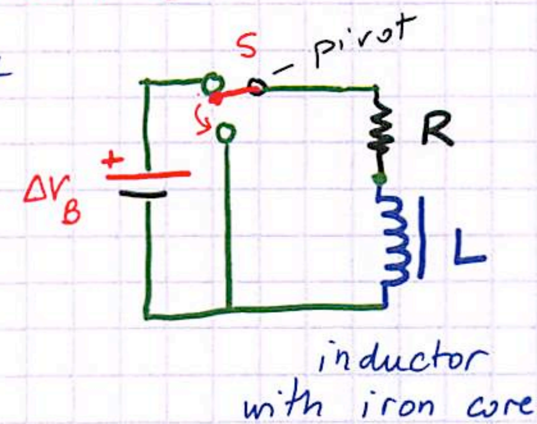
Electric fields $\vec{E}$ contain energy: when a capacitor with charge $\pm Q$ on its plates is connected to a resistor, then the discharge current heats the resistor → The electric energy associated with $\vec{E}$ is dissipated into heat.

Can we say something similar about the $\vec{B}$ field generated by a current passing through an inductor L?

Turning on a current is not easy: the inductance of a coil (or even a simple wire!) generates a counter-EMF which is overcome on the time scale of $\tau = L/R$ only. Here R is the combined resistance of the inductor (coil or wire) and the load.

What about turning off a current?

Long after ΔV_B was connected: the RL circuit is switched to a controlled discharge (as the RC was)



Idealized scenario: the switching is instantaneous (textbook discussion)

Kirchhoff loop law (S has closed in the discharge pos'n)

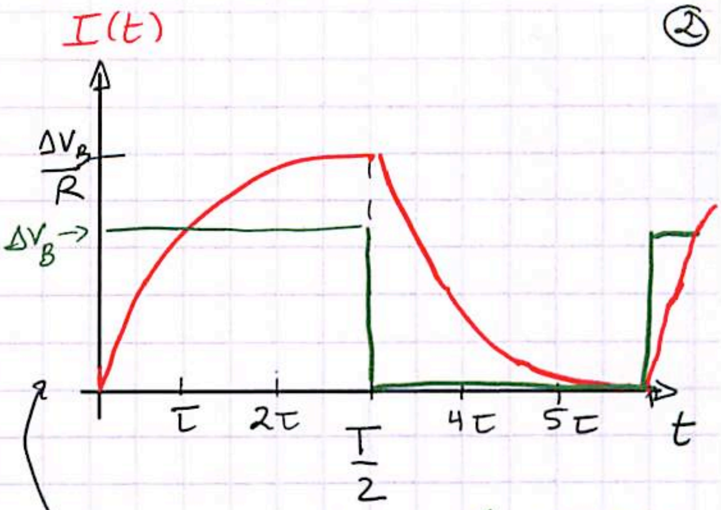
$$-R\,I(t) - L\frac{dI}{dt} = 0 \qquad I(t=0) = I_0 = \frac{\Delta V_B}{R}$$

battery was connected for many multiples of $\tau = L/R$ before S changed.

$$\therefore \quad I(t) = -\frac{L}{R}\frac{dI}{dt} = -\tau\frac{dI}{dt}$$

$$\therefore \quad I(t) = I_0\, e^{-t/\tau}$$

Thus, if we continuously switch the battery into the circuit and then replace it with a short (a square-wave generator, except that in the low state it is not a short, but a $50\,\Omega$ internal resistance), and the switching is on a timescale $T \gg \tau = \frac{L}{R}$, we see



— state of square-wave generator $0 \leftrightarrow \Delta V_B$

— current in the circuit as measured via the voltage drop across R

This looks very benign, and follows from the equations. It raises the question: what makes the current to continue to flow after the battery was disconnected?

The $\vec{B}$ field's inertia (according to Lenz/Faraday) is to blame. The term $-L\frac{dI}{dt}$ represents an EMF.

$\mathcal{E} = -\frac{d}{dt}\Phi_M$ where $\Phi_M = L\,I$ is now providing a <u>forward</u> EMF (since $\frac{dI}{dt} < 0$ for a decreasing current)

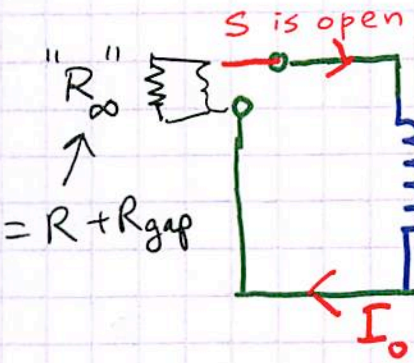
If the current were to stop suddenly, then $\frac{dI}{dt} = $ "$-\infty$", and a huge voltage would be generated by the inductor. Is this for real?

In fact, yes! When we use a simple battery to power an inductor, and then remove the contact by pulling the wire from a terminal $\longrightarrow$ sparks fly!

what does this mean?

Opening the contact (and not closing @ $t=0$ to make a short as discussed above)

creates an RL discharge circuit with $R \to \infty$.



Even though the circuit is broken, $I(t)$ has to continue to flow.

The gap between the switch contacts acts like a capacitor: the contact points acquire charge $\pm Q$, an $\vec{E}$ field builds up, and a discharge occurs.

Discharge: electrons ionize the air molecules $\to$ charges can support a current $\to$ R_∞ becomes finite

Q: is this a tiny capacitor or a big resistor??

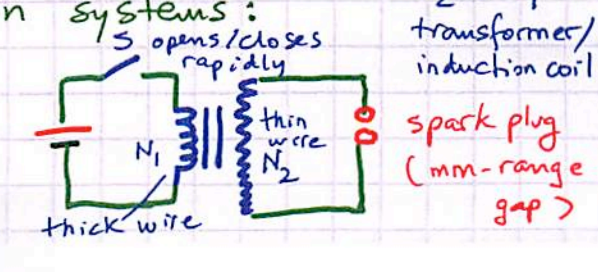
Practical (electronic/electrical engineering) messages:

1) switching off currents in large inductors (e.g., motors) creates transient voltages $\mathcal{E} = -L\frac{dI}{dt}$ in the thousands of volts range (otherwise no sparking / arcing!)

2) switches will burn $\to$ need protective circuits, capacitor across contact points
↳ allows transient current to flow

3) power grid, electronic power supplies $\to$ electrically dirty
$\to$ protect sensitive circuits by "voltage clamps"

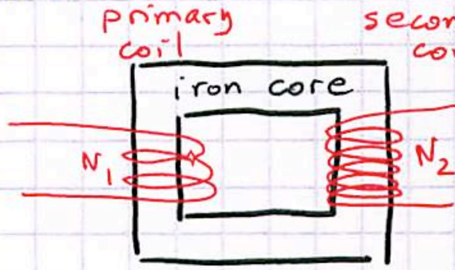
4) "pre-electronic" car ignition systems:
use a rapidly switching primary inductor coil to generate high-voltage peaks:

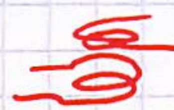


Transformers and AC circuits

Faraday's law $\mathcal{E} = -\frac{d\Phi_M}{dt} = -L\frac{dI}{dt}$ opens up interesting possibilities for non-constant electric currents.

In a transformer



We feed the primary coil with a sinusoidal (or some other periodic) current and generate a time-varying magnetic flux $\Phi_M(t)$ which is guided by the iron core into the secondary coil. Each of the N_2 secondary coil turns now generates $\mathcal{E} = -\frac{d\Phi_M}{dt}$; when added in series one realizes that: (ignoring the core )

@ primary coil: $\Delta V_{AC_1} = -\frac{d\Phi_M}{dt}$ — N_1 turns produce a time-varying magnetic flux $\Phi_M(t)$

each turn is driven by $\frac{1}{N_1}\Delta V_{AC}$!

@ secondary coil in each of the N_2 turns:

$$\frac{d\Phi_M}{dt} \Rightarrow \mathcal{E} \quad \therefore \quad \text{in series: } N_2\,\mathcal{E} \text{ is produced in total}$$

$$\Delta V_{AC_2} = N_2\frac{d\Phi_M}{dt} = N_2\frac{\Delta V_{AC_1}}{N_1} \qquad \therefore \Delta V_{AC_2} = \frac{N_2}{N_1}\Delta V_{AC_1}$$

1) We can step up/down time-varying voltages by adjusting the number of turns in the primary and secondary coils
2) Ohmic resistance in the primary causes some heating loss
3) under load: more heat is produced as the current load increases

Energy (density) associated with $\vec{B}$ (extra)

For an $\vec{E}$ field in a capacitor: $PE = \frac{1}{2} C \Delta V^2$; $E = \frac{\Delta V}{d}$

Q: how is PE stored in a $\vec{B}$ field?

A: We derived the "charge" cycle for an RL circuit $(B \sim I)$

$$I(t) = \frac{\Delta V_B}{R}\left(1 - e^{-t/\tau}\right) \qquad \tau = \frac{L}{R}, \quad \frac{\Delta V_B}{R} = I_{max}$$

The voltage drop across the inductor $\Delta V_L = -L \frac{dI}{dt}$

$$\Delta V_L(t) = -L \frac{\Delta V_B}{R}\left(+\frac{1}{\tau} e^{-t/\tau}\right) = \underbrace{-\Delta V_B \, e^{-t/\tau}}_{\text{initially } -\Delta V_B}$$

eventually $\longrightarrow 0$!

The power associated with loading the inductor:

$$P_L = |\Delta V_L| \cdot I = \frac{\Delta V_B^2}{R}\Big(\underbrace{e^{-t/\tau} - e^{-2t/\tau}}_{\geq 0\,!}\Big) \quad \begin{array}{l} \to @\, t=0 \;\; = 0 \\ \to @\, t \to \infty \;\; = 0 \end{array}$$

We need: PE for $t \to \infty$ when $I = I_{max} = \frac{\Delta V_B}{R}$

and $B = B_{max}$

$$P_L = \frac{\Delta PE}{\Delta t} \qquad \therefore \quad PE = \int_0^\infty P_L(t)\,dt$$

$$\int_0^\infty e^{-t/\tau} dt = -\tau e^{-t/\tau}\Big|_0^\infty = \tau \; ; \quad \int_0^\infty e^{-2t/\tau} dt = \frac{\tau}{2}$$

$$\therefore \quad PE_L = \frac{\Delta V_B^2}{R}\left(\tau - \frac{\tau}{2}\right) = \frac{1}{2}\frac{\Delta V_B^2}{R}\left(\frac{L}{R}\right) = \frac{1}{2} L \frac{\Delta V_B^2}{R^2} = \frac{L}{2} I_{max}^2$$

Thus, we found the counter-part to the energy stored in a capacitor:

$$PE_L = \frac{1}{2} L \cdot \underbrace{I^2}_{\text{steady-state current}}$$

Given that the magnetic field strength $B \sim I$,

we have: $PE_L \sim B^2$ in analogy to $PE_C \sim E^2$

In detail: $$PE_L = \frac{1}{2} L \left(\frac{B d}{\mu_0 N}\right)^2 = \frac{1}{2} \underbrace{\left(\frac{\mu_0 N^2 A}{d}\right)}_{L_{sol.}} \frac{d^2}{\mu_0^2 N^2} B^2 = \frac{B^2}{2\mu_0} \underbrace{(Ad)}_{\text{solenoid volume}}$$

$B_{sol} = \mu_0 \frac{NI}{d}$