# Rotational Energy

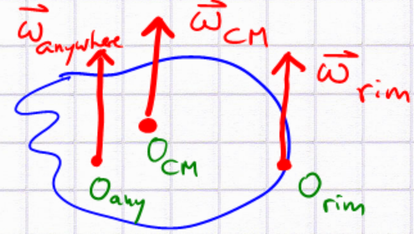
When an extended solid object is in general motion, i.e., CM in translation & object rotates about the CM the total kinetic energy has two additive contributions:

$KE_{tr} = \frac{1}{2} M v_{CM}^2$ is an obvious part

To understand the rotational part (presumably $\frac{1}{2} I \omega^2$ from translation table) we first need to understand that:

$\omega$ = rotation rate (angular velocity)

is the same for all mass elements $m_i$ that make up the body.



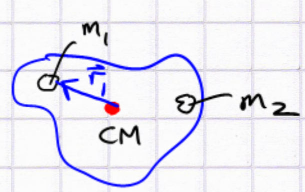

Why?

different choices of pivot point O result in the CM carrying out a different trajectory, but the rotation (spin) of the object remains the same!

coin → Maple Leaf turns at some rate given by $\omega = xx \frac{rad}{sec}$.

∴ All mass elements $m_i$ experience the same $\omega$.

Consider the CM as pivot point



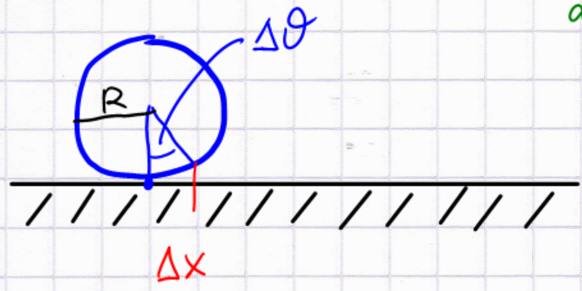

Look at the KE of $m_1$:

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 (r_1 \omega)^2 = \frac{1}{2} m r_1^2 \omega^2$$

- analogously for $m_2, m_3, \dots, m_N$ → entire body

$$KE_{rot} = \sum_i \left(\frac{1}{2} m_i v_i^2\right) = \sum \left(\frac{1}{2} m r_i^2\right) \omega^2 = \frac{1}{2} \underbrace{\left(\sum m_i r_i^2\right)}_{\text{inertia about CM}} \omega^2 = \frac{1}{2} I \omega^2$$

Now understand rolling




as the wheel rotates by $\Delta\vartheta$

the CM moves forward by $\Delta x$

$$\Delta x = R\,\Delta\vartheta \qquad \text{divide by } \Delta t$$

$$v = R\omega$$

Assuming there is no slipping : $v = v_{CM}$ is locked with $\omega$

$$KE_{tot} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I \left(\frac{v_{CM}^2}{R^2}\right)$$

For a disk (solid wheel) $I_{CM} = \frac{1}{2} MR^2$

$$\therefore \quad KE_{tot} = \frac{1}{2} M v_{CM}^2 + \frac{1}{4} M v_{CM}^2 = \frac{3}{4} M v_{CM}^2$$

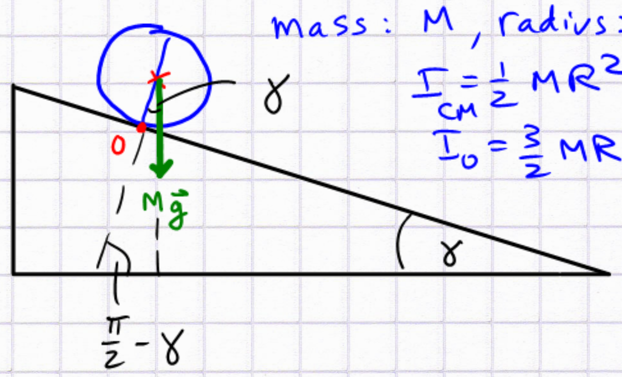
$\therefore$ slipping costs less energy!

Rolling on an incline :

mass: M, radius: R

$I_{CM} = \frac{1}{2} MR^2$

$I_0 = \frac{3}{2} MR^2$




Remember: slipping $\rightarrow$

$$a_{CM} = g \sin\gamma$$

$$\tau_0 = -RMg\sin\gamma \qquad (\text{why? lever arm} = 0 \rightarrow CM)$$

$$\sin(\gamma) = \sin(\pi - \gamma)$$

Friction @ 0 $\rightarrow$ no torque (zero arm length)

$$I_0\,\alpha = -RMg\sin\gamma \rightarrow \text{neg sign} \rightarrow \text{CW rotation}$$

$$\frac{3}{2} MR^2 \alpha = -RMg\sin\gamma \quad \therefore \alpha = \frac{2}{3}\frac{g}{R}\sin\gamma \quad \therefore a_{CM} = R\alpha = \frac{2}{3} g\sin\gamma$$

Now we understand: slipping motion $\rightarrow$ $a_{CM} = g \sin\gamma$

rolling disk $\rightarrow$ $a_{CM} = \frac{2}{3} g \sin\gamma$

How can we understand the latter result?

1) Gravitational PE $V_g = Mg\,\Delta y$ is converted partly into translational, partly into rotational KE

2) The motion of the CM can be understood by applying the NET force at the CM.

Gravity $\longrightarrow$ $Mg\sin\gamma$

Friction $\vec{f}_s$ @ pivot O $\longrightarrow$ ?

$$M a_{CM} = Mg\sin\gamma - f_s$$

(assume $\vec{f}_s$ acts to oppose $\vec{g}_{\parallel}$)

$$\therefore \quad f_s = M\left(g\sin\gamma - a_{CM}\right)$$

$$= M\left(g\sin\gamma - \tfrac{2}{3} g \sin\gamma\right) = -\frac{M}{3} g\sin\gamma$$

Perfect rolling motion (no slip):

$\vec{f}_s$ adjusts itself (as long as $|\vec{f}_s| \le \mu_s N$) such as to ensure rolling (point of contact = O @ rest)

$\rightarrow$ analogous to: try to push a heavy object $\rightarrow$ doesn't move

$\rightarrow$ microscopic view: electrostatic forces between floor and object try to hold O stationary