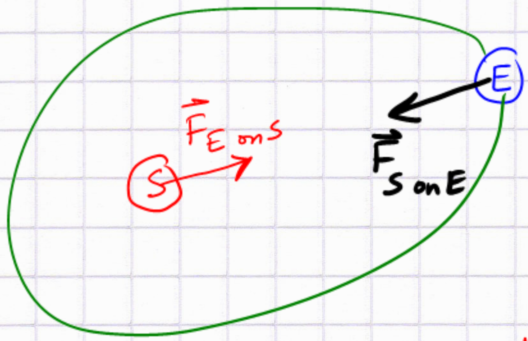
Angular Momentum Theorem → application

Consider a simplified planetary system:

sun + earth (no moon, no other planets)



an elliptic (near-circular) orbit

$$\vec{F}_{S\,\text{on}\,E} = -\frac{G M_E M_S}{r^3}\vec{r}$$

consider the sun so massive, that even though $|\vec{F}_{E\,\text{on}\,S}| = |\vec{F}_{S\,\text{on}\,E}|$ the sun's acceleration towards the earth is negligible.

The earth is torque-free (using S as the origin for the rotation axis) since $\vec{r}$ and $\vec{F}_{S\,\text{on}\,E}$ are collinear ($\alpha = \pi$ and $\sin(\alpha) = 0 \Rightarrow \tau_z = r F_{S\,\text{on}\,E} \sin\alpha = 0$)

Thus, L_z = constant in time → Kepler's area law
p. 293-4 in Giordano.

An equally important, more sophisticated implication:

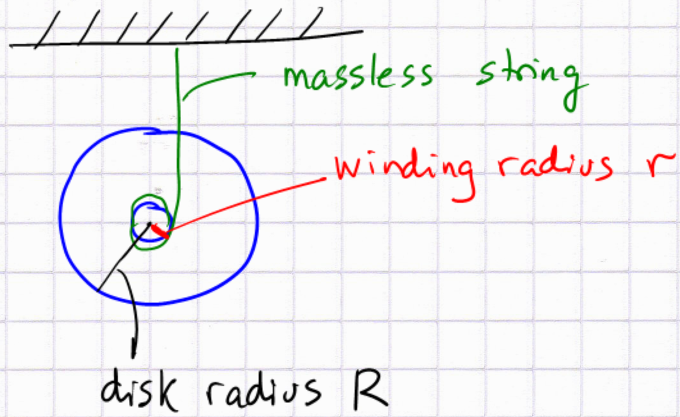
$$\frac{d}{dt}\vec{L}_{earth} = \vec{0} \qquad \vec{L}_{earth} = \vec{r} \times \vec{p} = \vec{L}_0 \text{ (= const. vector)}$$

The direction of $\vec{L}_{earth}$ doesn't change, the motion is truly planar. → A consequence of a central force, i.e., a force pointing towards the sun.

In reality: the other planets tug at earth, this is a non-central contribution ⇒ the plane has <u>some wobble</u> over longtimes!

- Back to rotations about a fixed axis (test 3 prep)

yo-yo



(assumed to be constant → idealization)

$r < R$

Want to understand: why does the yo-yo fall so slowly?

motion of the CM

Two physics principles:

1) Motion of the CM → $M\vec{g}$ pulls down → $\vec{T}$ tension force pulls up

① $M a_{CM} = Mg - T$ chose $\downarrow \hat{j}$

- can't solve, as we don't know T.

2) Rotational Motion → about CM?

↳ about point where the string touches?

we learned: $\omega_{CM} = \omega_0$ ← rotation about a pivot point

but let's do it about CM.

$I_{CM} = \frac{1}{2} M R^2$ (yo-yo is basically 2 disks)

$I_{CM} \alpha = \tau_{net}$ ← given by the tension, since gravity has zero arm length

Given how we drew it : CCW rotation $\longrightarrow$

$$\frac{1}{2} M R^2 \alpha = r T$$

($\vec{T}$ is up, $\vec{r}$ is to the right ($+x$), $\tau_z > 0$)

2 $$\alpha = \frac{2 r T}{M R^2}$$

r is the winding radius !

3) Connection : translation of CM $\longleftrightarrow$ rotation about CM

it is the winding radius r that counts here !

$$r \, \Delta\vartheta = \Delta y \quad \therefore \quad \omega = \frac{v_y}{r} \quad \therefore \quad \alpha = \frac{a_y}{r} \quad (3)$$

$a_y \equiv a_{CM}$

Now solve : 3 inserted into 2 $\longrightarrow$

$$\frac{a_{CM}}{r} = \frac{2 r T}{M R^2}$$

solve for T + insert in 1

$$T = \frac{M}{2} \frac{R^2}{r^2} a_{CM}$$

1 $$M a_{CM} = M g - \frac{M}{2} \frac{R^2}{r^2} a_{CM}$$

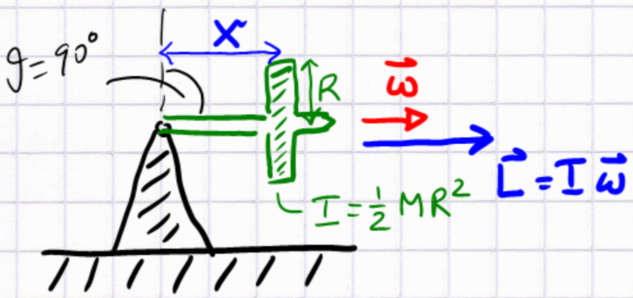
$$a_{CM} = g - \frac{1}{2} \frac{R^2}{r^2} a_{CM}$$

$$a_{CM} \left(1 + \frac{R^2}{2 r^2} \right) = g \quad \therefore \quad a_{CM} = g \, \frac{2 r^2}{2 r^2 + R^2}$$

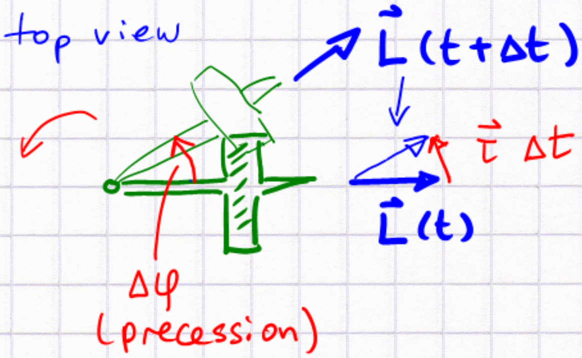
For $r << R$ this is a small translational acceleration !

Back to the gyroscope $\rightarrow$ use the AM theorem to understand the increase in precession rate as the gyro slows down

side view



top view



$$\Omega = \frac{\Delta\varphi}{\Delta t}$$

precessional motion: $\vec{L}(t)$ moves on a circular path $\rightarrow$

- compare to $\vec{R}(t)$ in uniform circular motion
- what connects $\vec{R}(t+\Delta t)$ and $\vec{R}(t)$?

$$\vec{R}(t+\Delta t) = \vec{R}(t) + \vec{V}(t)\,\Delta t$$

$$\left(\text{follows from} \quad \vec{V}(t) = \frac{d}{dt}\vec{R} \right)$$

- in uniform motion $\quad \vec{R}(t) = R_0 (\hat{\imath} \cos\Omega t + \hat{\jmath} \sin\Omega t)$

$(R_0 = |\vec{R}(t)|) \rightarrow \quad \vec{V}(t) = R_0 \Omega (-\hat{\imath} \sin\Omega t + \hat{\jmath} \cos\Omega t)$

$$V = R_0 \Omega \qquad \text{or} \qquad \Omega = \frac{V}{R_0}$$

By analogy: $\quad \vec{L}(t) = L_0 (\hat{\imath} \cos\Omega t + \hat{\jmath} \sin\Omega t)$

$$\frac{d\vec{L}}{dt} = L_0 \Omega (-\hat{\imath} \sin\Omega t + \hat{\jmath} \cos\Omega t)$$

$$\left| \frac{d\vec{L}}{dt} \right| = L_0 \Omega \quad , \quad \text{but} \quad \frac{d\vec{L}}{dt} = \vec{\tau} \; ; \; L_0 = I\omega$$

$$L_0 \Omega = \underbrace{M g X}_{\tau} \quad \therefore \quad \underline{\underline{\Omega}} = \frac{MgX}{L_0} = \frac{MgX}{\frac{1}{2}MR^2\omega} = \underline{\underline{\frac{2Xg}{R^2\omega}}}$$

blows up as $\omega \rightarrow 0$!