Angular Momentum Theorem → application

Consider a simplified planetary system:

sun + earth (no moon, no other planets)

an elliptic (near-circular) orbit

\[ \vec{F}_{\text{sun}E} = -\frac{G M_s M_e}{r^3} \vec{r} \]

consider the sun so massive, that even though \( |\vec{F}_{\text{Ems}}| = |\vec{F}_{\text{sinE}}| \) the sun’s acceleration towards the earth is negligible.

The earth is torque-free (using \( \mathcal{S} \) as the origin for the rotation axis) since \( \vec{r} \) and \( \vec{F}_{\text{sun}E} \) are collinear

\( \alpha = \pi \) and \( \sin (\alpha) = 0 \Rightarrow L_z = r \vec{F}_{\text{sun}E} \sin \alpha = 0 \)

Thus, \( L_z = \text{constant in time} \rightarrow \text{Kepler's area law} \)

An equally important, more sophisticated implication:

\[ \frac{d}{dt} \vec{L}_{\text{earth}} = 0 \quad \vec{L}_{\text{earth}} = \vec{r} \times \vec{p} = \vec{L}_0 \quad (\text{const. vector}) \]

The direction of \( \vec{L}_{\text{earth}} \) doesn’t change, the motion is truly planar. → A consequence of a central force, i.e., a force pointing towards the sun.

In reality: the other planets tug at earth; this is a non-central contribution ⇒ the plane has some wobble over long times!
Back to rotations about a fixed axis (test 3 prep)

yo-yo

massless string

winding radius \( r \) (assumed to be constant \( \Rightarrow \) idealization) \( r < R \)

disk radius \( R \)

Want to understand: why does the yo-yo fall so slowly?

motion of the CM

Two physics principles:

1) Motion of the CM

\[ M \ddot{a}_{CM} = Mg - T \]

2) Rotational Motion \( \rightarrow \) about CM?

\[ I_{CM} \alpha = T \hat{r} \]

\( \hat{r} \) about point where the string touches?

we learned: \( \omega_{CM} = \omega_0 \) \( \leftarrow \) rotation about a pivot point

but let’s do it about CM.

\[ I = \frac{1}{2} MR^2 \] (yo-yo is basically 2 disks)

\( I_{CM} \alpha = T \hat{r} \) \( \leftarrow \) given by the tension, since gravity has zero arm length
Given how we drew it: CCW rotation $\Rightarrow$

$$\frac{1}{2} MR^2 \alpha = rT$$

(\(\vec{T}\) is up, \(\vec{r}\) is to the right (+x), \(T > 0\))

\(r\) is the winding radius!

2) Connection: translation of CM $\leftrightarrow$ rotation about CM

It is the winding radius \(r\) that counts here!

$$r \Delta \theta = \Delta y$$

\[\therefore \omega = \frac{\Delta y}{r}\] \(\therefore \alpha = \frac{\Delta y}{r}\)

\[\therefore \alpha_y = a_{CM}\]

Now solve: 3) inserted into 2) $\Rightarrow$

$$\frac{a_{CM}}{r} = \frac{2 rT}{MR^2}$$

Solve for \(T\) + insert in 1)

$$T = \frac{M}{2} \frac{R^2}{r^2} a_{CM}\]

1) \(Ma_{CM} = Mg - \frac{M}{2} \frac{R^2}{r^2} a_{CM}\]

$$a_{CM} = g - \frac{1}{2} \frac{R^2}{r^2} a_{CM}\]

$$a_{CM} (1 + \frac{R^2}{2r^2}) = g\] \(\therefore a_{CM} = g \frac{2r^2}{2r^2 + R^2}\]

For \(r \ll R\) this is a small translational acceleration!
Back to the gyroscope — use the AM theorem to understand the increase in precession rate as the gyro slows down.

Precessional motion: $\mathbf{L}(t)$ moves on a circular path —

- compare to $\mathbf{R}(t)$ in uniform circular motion
- what connects $\mathbf{R}(t+\Delta t)$ and $\mathbf{R}(t)$?

$$\mathbf{R}(t+\Delta t) = \mathbf{R}(t) + \mathbf{V}(t) \Delta t$$

( follows from $\mathbf{V}(t) = \frac{d}{dt} \mathbf{R}$ )

- in uniform motion $\mathbf{R}(t) = R_0 \left( \hat{i} \cos \omega t + \hat{j} \sin \omega t \right)$

$(R_0 = |\mathbf{R}(t)|)$  $\Rightarrow$  $\mathbf{V}(t) = R_0 \Omega \left( -\hat{i} \sin \omega t + \hat{j} \cos \omega t \right)$

$\mathbf{V} = R_0 \Omega$  or  $\Omega = \frac{\mathbf{V}}{R_0}$

By analogy:  $\mathbf{L}(t) = L_0 \left( \hat{i} \cos \omega t + \hat{j} \sin \omega t \right)$

$$\frac{d\mathbf{L}}{dt} = L_0 \Omega \left( -\hat{i} \sin \omega t + \hat{j} \cos \omega t \right)$$

$|\frac{d\mathbf{L}}{dt}| = L_0 \Omega$,  but $\frac{d\mathbf{L}}{dt} = \mathbf{e}$;  $L_0 = I \omega$

$$L_0 \Omega = \frac{MgX}{\ell}  \therefore  \Omega = \frac{MgX}{L_0} = \frac{MgX}{\frac{1}{2} MR^2 \omega} = \frac{2Xg}{R^2 \omega}$$

blows up as $\omega \to 0$!