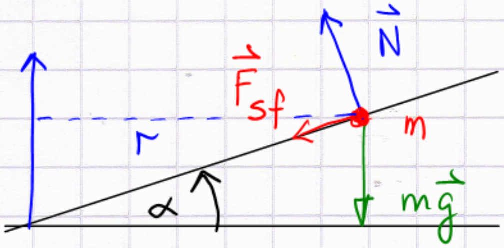
Example 5.3 Banked turn with friction (Giordano, College Physics, p. 137)



$\overleftarrow{\vec{a}_{cp}}$ $\quad \hat{j} \uparrow \rightarrow \hat{i}$

$a_y = 0$

$\therefore \quad N\cos\alpha = mg + F_{sf}\sin\alpha$

We are considering the maximum speed, just before F_{sf} reaches the maximally allowed value $\therefore \quad F_{sf} \overset{(<)}{=} \mu_s N$

Thus,

$$N\cos\alpha = mg + \mu_s N \sin\alpha \qquad \therefore \quad N = \frac{mg}{\cos\alpha - \mu_s \sin\alpha}$$

Now the $\hat{i}$ component:

$$\therefore \quad m a_{cp} = F_{sf}\cos\alpha + N\sin\alpha$$

$$= \mu_s N \cos\alpha + N\sin\alpha$$

$$\frac{\cancel{m}v^2}{r} = \frac{\cancel{m}g}{\cos\alpha - \mu_s\sin\alpha}\left(\mu_s\cos\alpha + \sin\alpha\right)$$

$$\boxed{v^2 = gr\,\frac{\mu_s\cos\alpha + \sin\alpha}{\cos\alpha - \mu_s\sin\alpha}}$$

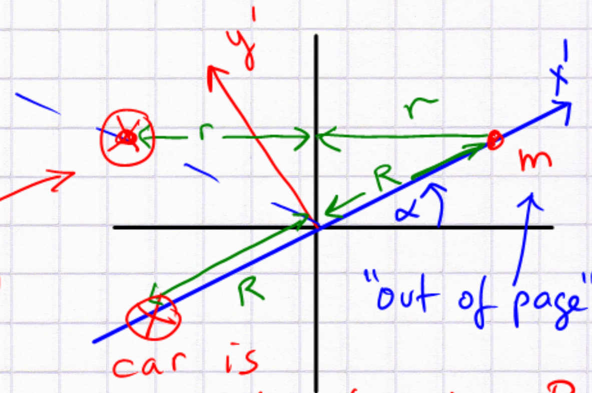
This is the textbook result. Can we obtain it also in a different coordinate system $\hat{j}' \uparrow \nearrow \hat{i}'$ where $\hat{i}$ is aligned with the road?

Only, if we are careful!

The car executes a circle of radius r about y ($\hat{j}$) in the above picture; not about $\hat{j}'$!!

Pretend we are rotating about $\hat{j}'$, which looks the same for the instant shown in Fig. 5.8

However, after 180° the car is at ⊗ on the left, not at ⊗, where it should be for a banked curve!



on a circle of radius R

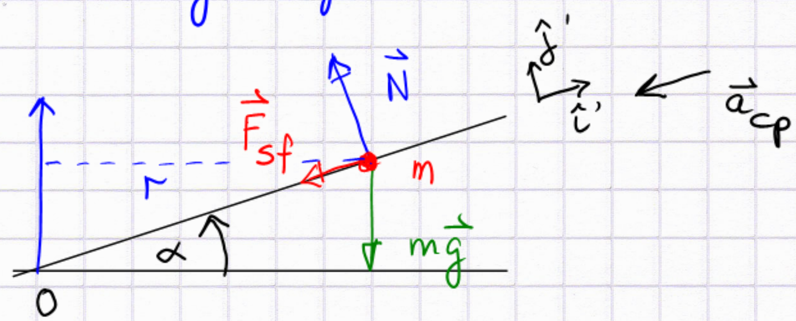
$\underbrace{\hspace{6em}}$

uniformly tilted surface

For the uniformly tilted surface (defined by $\hat{i}', \hat{j}'$) rotation about O with radius R

$$\frac{r}{R} = \cos\alpha$$

Free-body diagram:



Now we have: $\vec{F}_{sf} \sim -\hat{i}'$, $\vec{N} \sim \hat{j}'$

$\hat{j}'$ components: $N = mg\cos\alpha$

$\hat{i}'$ components:
$$m\,a_{cp} = F_{sf} + mg\sin\alpha$$
$$m\frac{v^2}{R} = \mu_s N + mg\sin\alpha$$
$$\cancel{m}\frac{v^2}{R} = (\mu_s\cos\alpha + \sin\alpha)\cancel{m}g$$

To compare with the previous result (banked curve)

$$v^2 = Rg\,(\mu_s \cos\alpha + \sin\alpha)$$

$$\boxed{v^2 = \frac{rg}{\cos\alpha}\,(\mu_s \cos\alpha + \sin\alpha)}$$

vs:

$$\boxed{v^2 = gr\,\frac{\mu_s \cos\alpha + \sin\alpha}{\cos\alpha - \mu_s \sin\alpha}}$$

The difference: denominator: $\cos\alpha \to \cos\alpha - \mu_s \sin\alpha$

- For small α the difference is small
- For moderate α we get a larger max. speed $v = \sqrt{v^2}$ for banked curve (denominator is smaller)

vs tilted plane.

Q: What is the physical origin of this difference?

Note: Even though the radius r (banked curve) is smaller than R (tilted plane) $r = R\cos\alpha$

we can sustain a higher speed without skidding.

We have a larger normal force: $N = mg\cos\alpha$ in the tilted plane case is much less than what we found for the banked curve

$$N = \frac{mg}{\cos\alpha - \mu_s \sin\alpha}$$

What causes this dramatic difference in the magnitude of N?

In the case of the banked curve (circle with radius r about y-axis) - for the condition of max speed v (we should label it as v_{cr} just before skidding) the car presses more into the pavement

The free-body diagram leads to a different magnitude of N depending on whether we assume

$\overleftarrow{}$ $\vec{a}_{cp}$

banked curve
(Example 5.3)

$\swarrow$ $\vec{a}_{cp}$

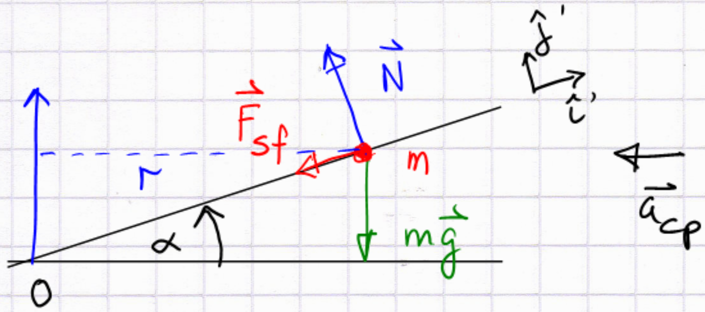
tilted plane
(misinterpreted Ex. 5.3)

The snapshot (Fig 5.8) is inconclusive without this additional information (or implied info)

Another reason why a FB diagram should include an indication of $\vec{a}$ given by $\vec{F}_{net}/m$.

Can we solve the banked curve in $\hat{j}' \nwarrow \nearrow \hat{i}'$ coordinates? It's not convenient, since $\vec{a}_{cp}$ has components $-a_{cp}\cos\alpha\ \hat{i}' + a_{cp}\sin\alpha\ \hat{j}'$

$\therefore$ 2 equations



$\hat{j}'$: $\quad m a_{cp} \sin\alpha = N - mg\cos\alpha$

$\hat{i}'$: $\quad m a_{cp}\cos\alpha = F_{sf} + mg\sin\alpha$

$\qquad\qquad\qquad = \mu_s N + mg\sin\alpha$

$$\therefore \quad N = mg\cos\alpha + m a_{cp}\sin\alpha$$

$$\cancel{m} a_{cp}\cos\alpha = \mu_s(\cancel{m}g\cos\alpha + \cancel{m}a_{cp}\sin\alpha) + \cancel{m}g\sin\alpha$$

$$a_{cp}(\cos\alpha - \mu_s\sin\alpha) = g(\mu_s\cos\alpha + \sin\alpha)$$

$$a_{cp} = \frac{g(\mu_s\cos\alpha + \sin\alpha)}{\cos\alpha - \mu_s\sin\alpha}$$

$$\frac{v^2}{r} = \frac{g(\mu_s\cos\alpha + \sin\alpha)}{\cos\alpha - \mu_s\sin\alpha}$$

$$v^2 = rg\,\frac{\mu_s\cos\alpha + \sin\alpha}{\cos\alpha - \mu_s\sin\alpha}$$

Which is the same result as derived in Example 5.3

$$\boxed{v^2 = gr\,\frac{\mu_s\cos\alpha + \sin\alpha}{\cos\alpha - \mu_s\sin\alpha}}$$