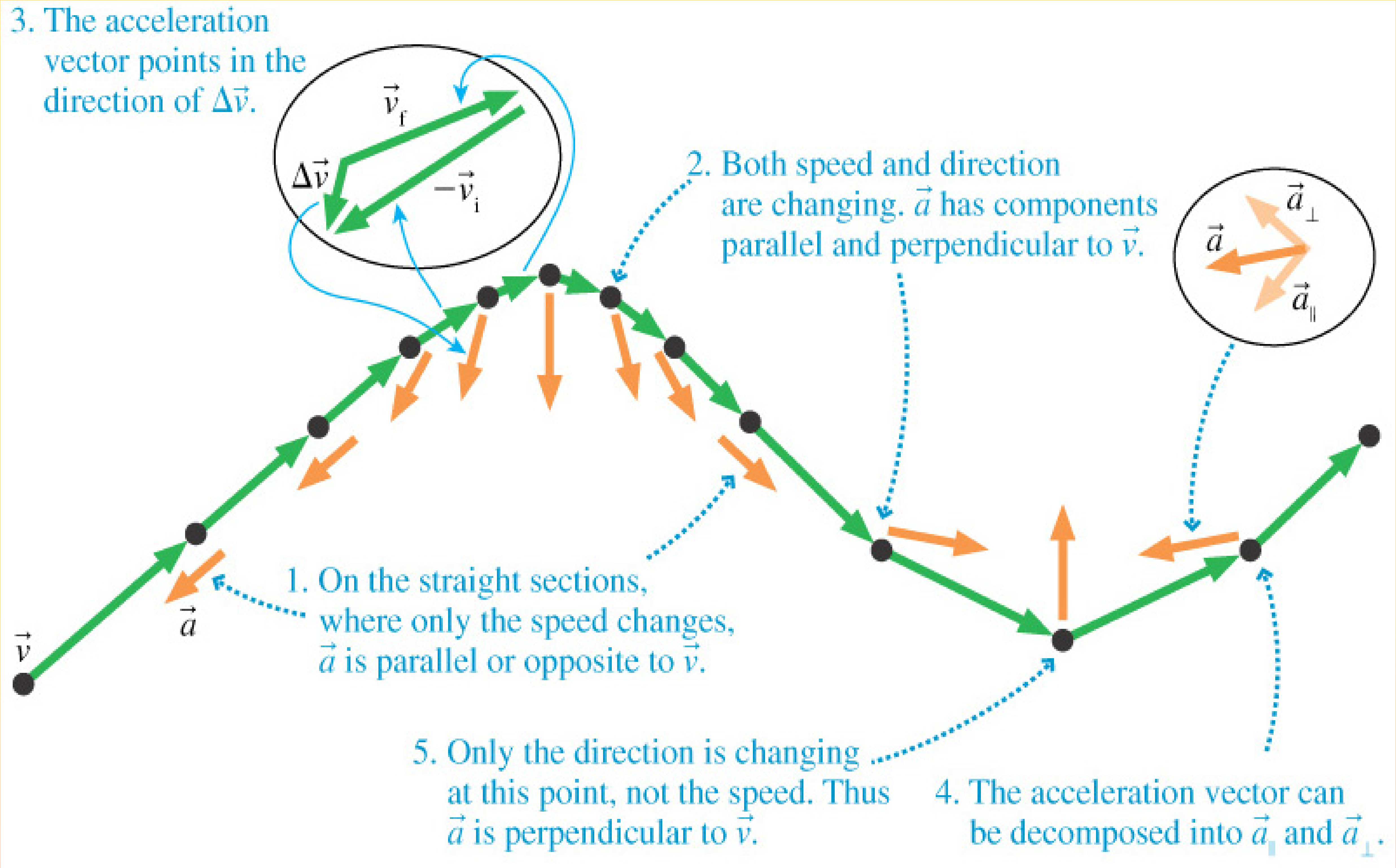
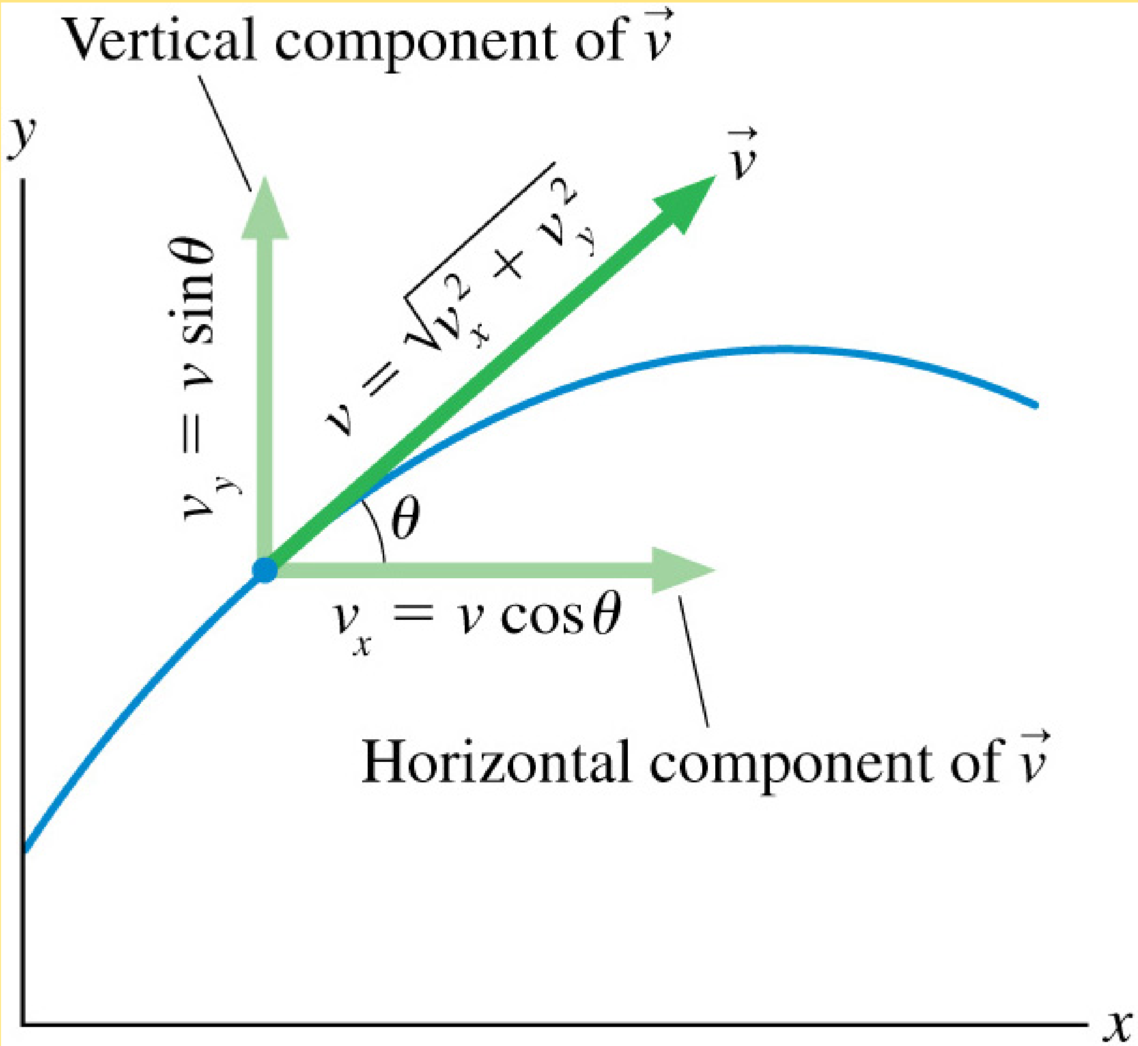
Motion in 2d

Understand the relation between $\vec{v}(t)$ and $\vec{a}(t)$ on a roller coaster. Is gravity the only force?



Vector Components



Standard decomposition:

$$\vec{v} = v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}}$$

Speed:

$$v(t) = \sqrt{v_x(t)^2 + v_y(t)^2}$$

Components:

$$v_x(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = v(t)\cos(\theta(t))$$

$$v_y(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = v(t)\sin(\theta(t))$$

Message:

The vectors are described by Cartesian components (v_x, v_y),
or by magnitude v and orientation angle θ (polar representation)

$$\tan(\theta(t)) = \frac{v_y(t)}{v_x(t)}$$

Newton's 2nd law in Cartesian coordinates:

- practical and straightforward
- sometimes not economical

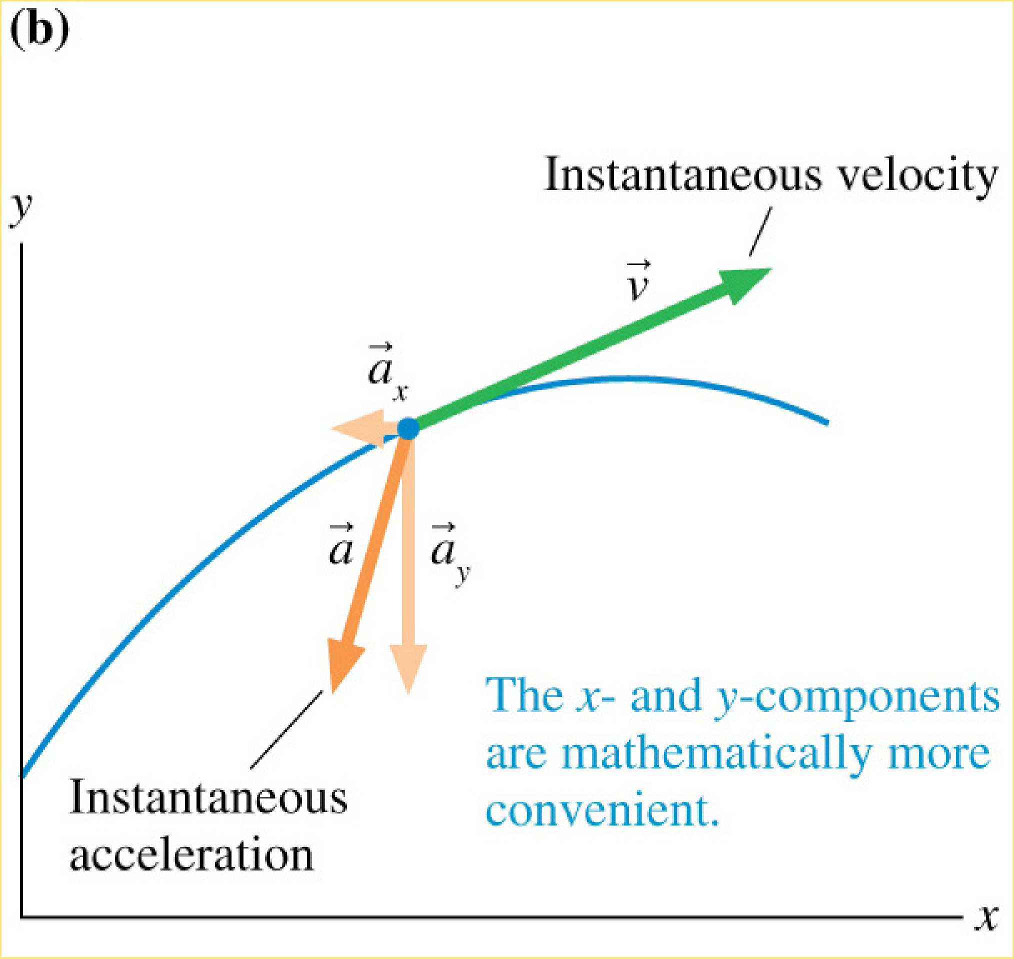
Understand Acceleration

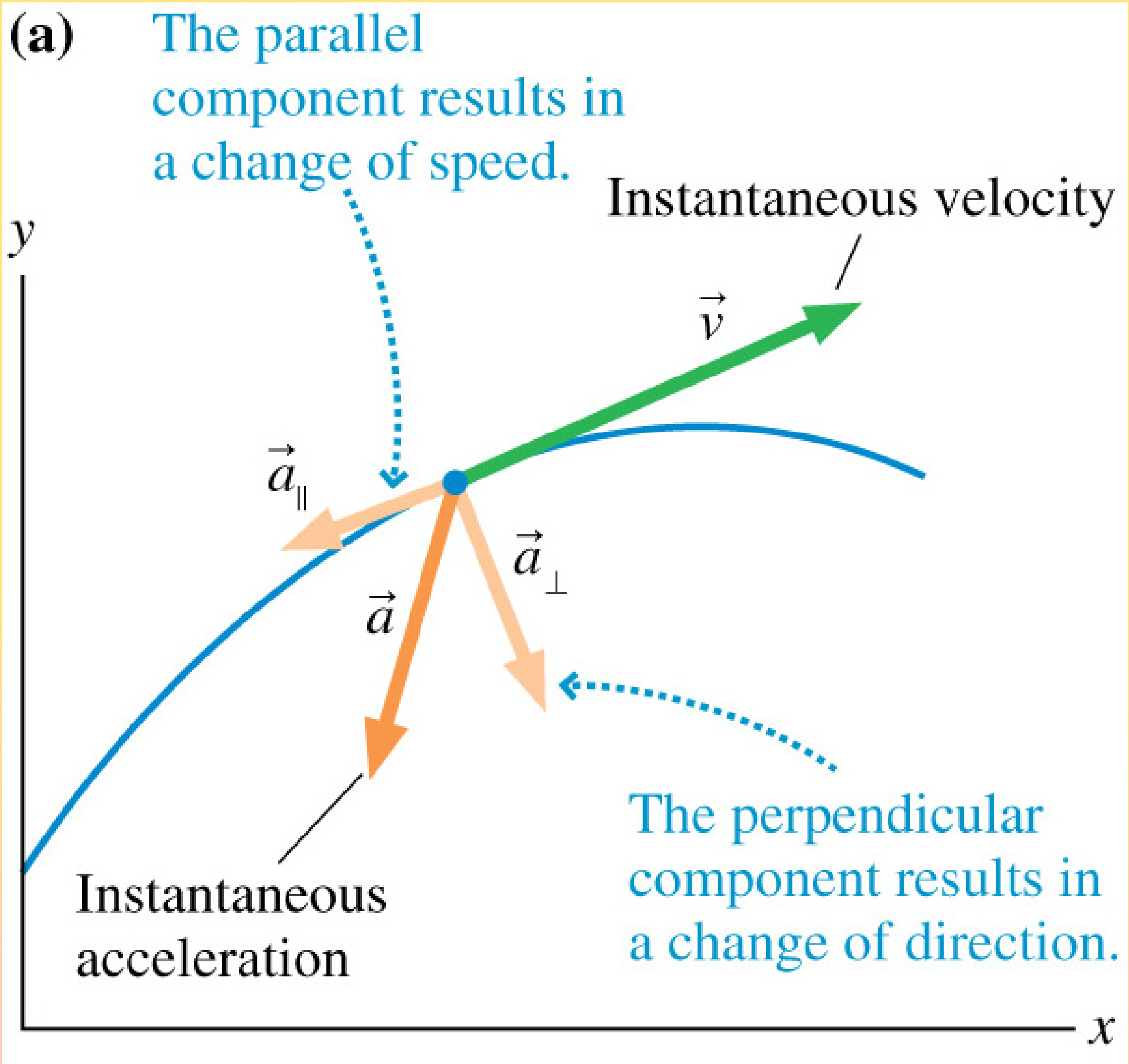
standard decomposition of $\vec{a}$

effect on of $\vec{v}$ not easily understood

would like to see parts responsible for

- change in speed
- change in orientation



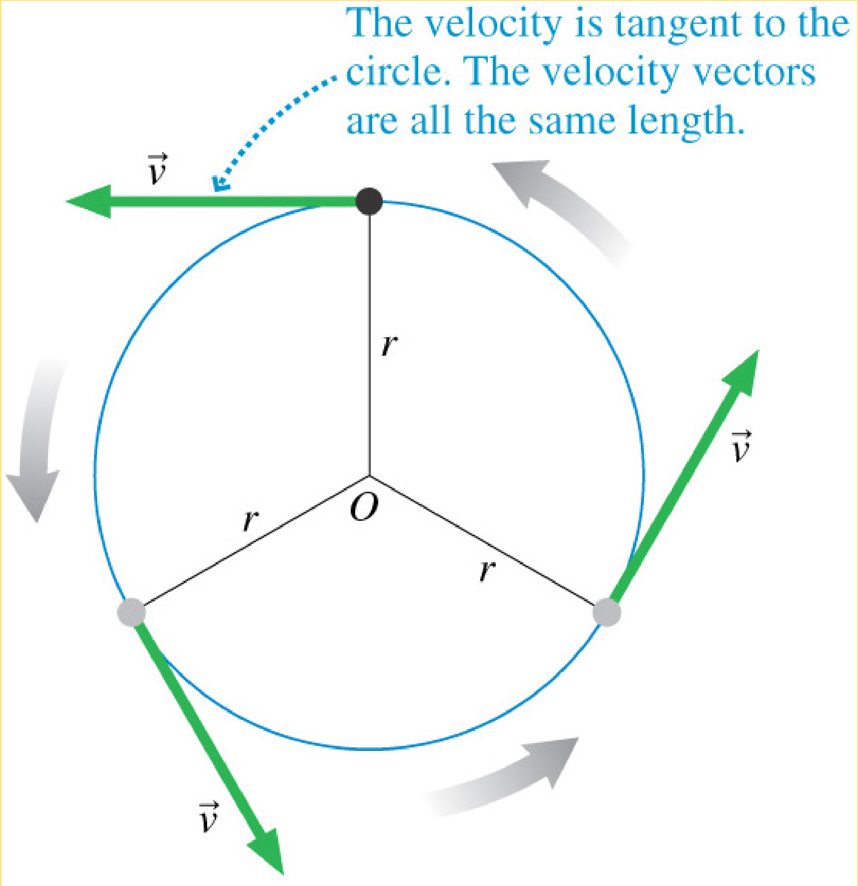


$$\vec{a} = \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j}$$

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$$

Guided motion (roller coaster): parallel/perpendicular force components is relevant

Uniform Circular Motion



Definitions:

Period T = time to go once around

Speed $v = \frac{2\pi r}{T}$

Revolution $1\text{rev} = 360° = 2\pi$ rad

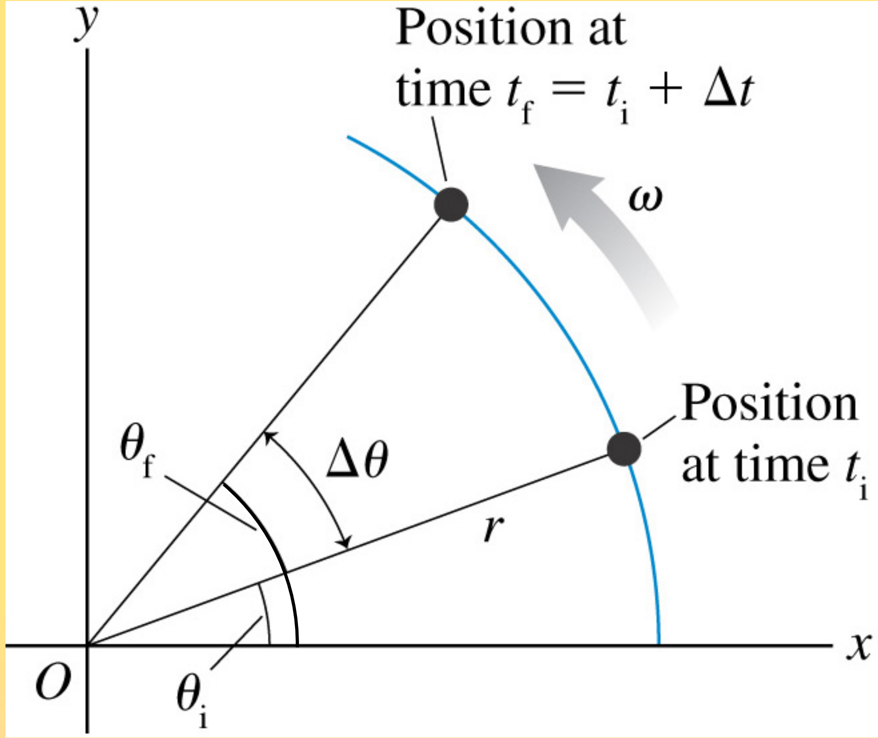
Radians $1\text{rad} = 1\text{rad} \times \frac{360°}{2\pi \text{ rad}} \approx 57.3°$

Arc length $s = r\theta$, when θ measured in rad

Angular position θ changes with time t

Angular velocity ω is constant in t

In general: $\omega \equiv \frac{d\theta}{dt}$ a function of time t



$x = r\cos\omega t$, $y = r\sin\omega t$, $\theta = \omega t$

position vector $\vec{r}(t) = ?$

$\vec{r}(t) = r\cos\omega t\,\hat{\mathrm{i}} + r\sin\omega t\,\hat{\mathrm{j}}$ $\quad \vec{v}(t) = ?$

$\vec{v}(t) = -r\omega\sin\omega t\,\hat{\mathrm{i}} + r\omega\cos\omega t\,\hat{\mathrm{j}}$ $\quad \vec{a}(t) = ?$

$\vec{a}(t) = -r\omega^2\cos\omega t\,\hat{\mathrm{i}} - r\omega^2\sin\omega t\,\hat{\mathrm{j}} = -\omega^2\vec{r}$ (!)

Calculate $v = |\vec{v}|$, $a = |\vec{a}|$

$v^2 = \omega^2 r^2$, i.e., $v = \omega r$; $\quad a = \omega^2 r = \frac{v^2}{r}$ centripetal acc.

Non-uniform Circular Motion

Use of planar vectors - overkill ?

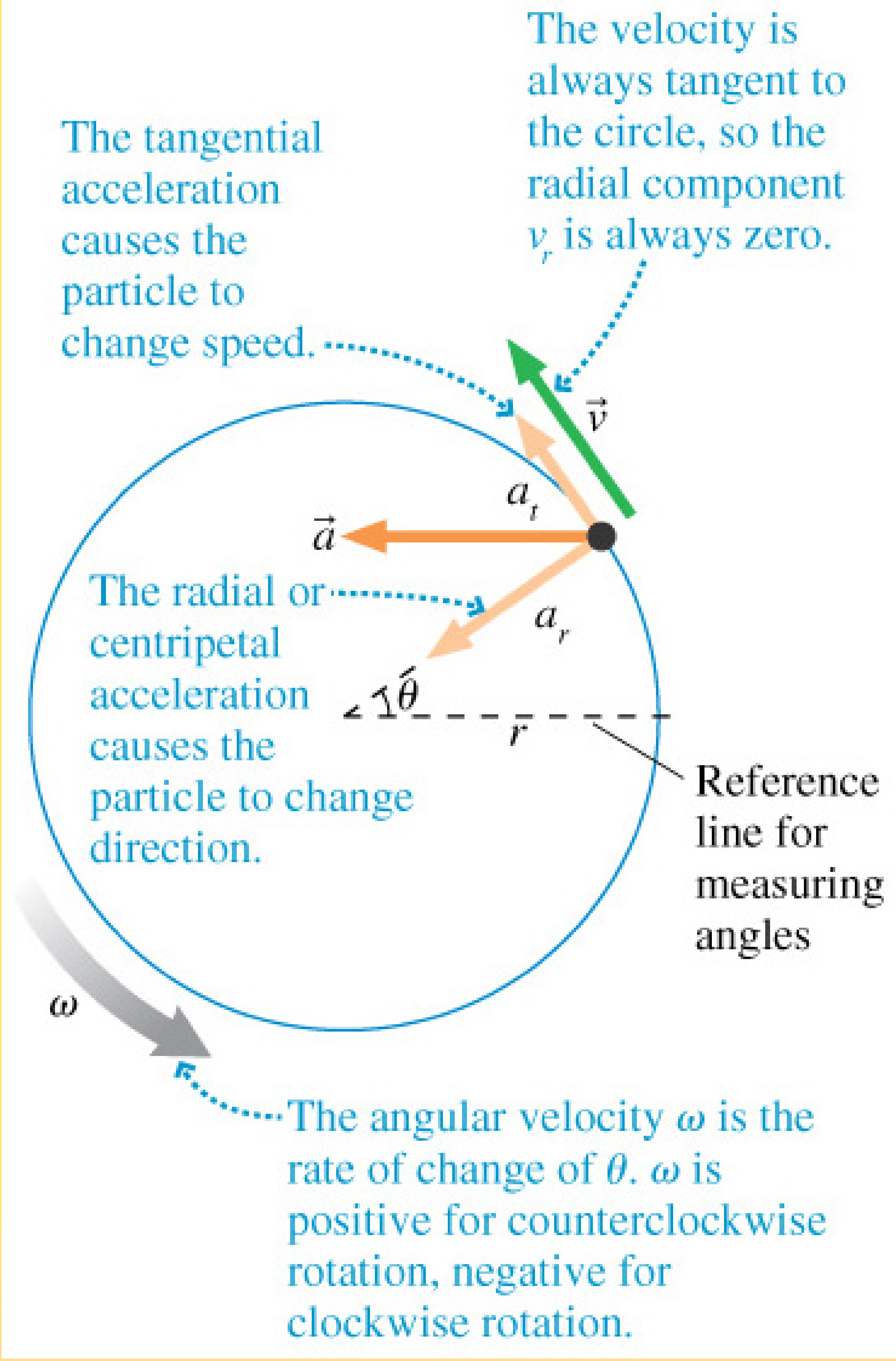
One-dimensional approach ?

$\theta(t)$, $\omega(t) = \theta'(t)$, $\alpha(t) = \omega'(t) = \theta''(t)$

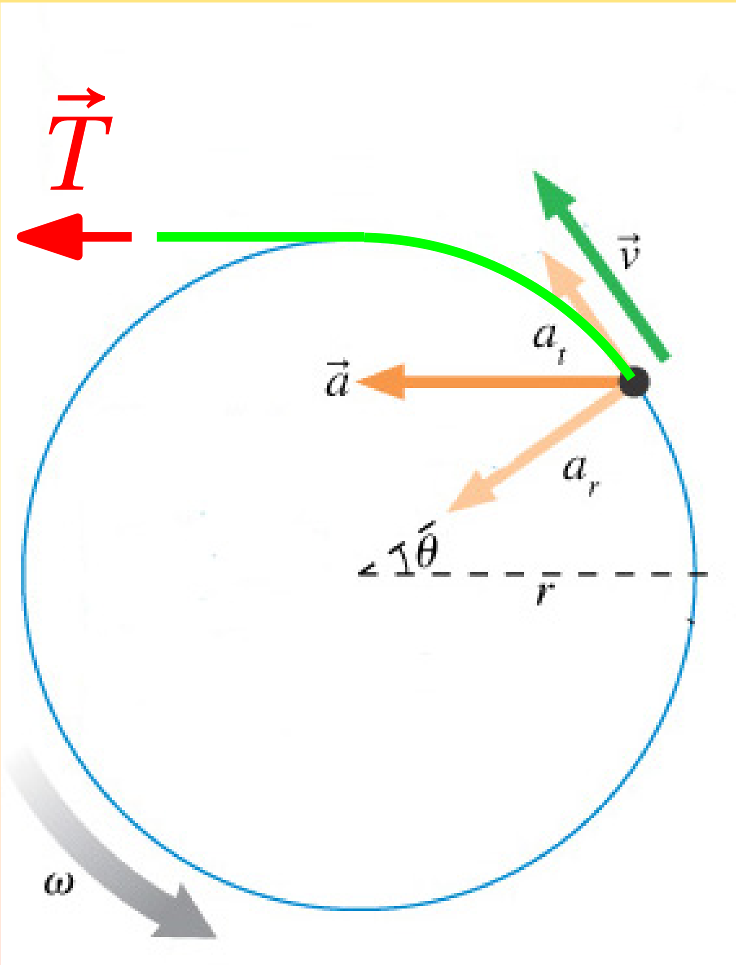
Angular acceleration $\alpha(t)$

(constant, for now, i.e., $\alpha(t) = \alpha_0$)

String tension force $\vec{T}$ provides α_0



String is wound at radius r



spool's mass ignored compared to m

How does $\vec{T}$ accelerate m ?

$$a_t = v'(t) = \frac{T}{m}$$

How does $\omega = v/r$ change ?

$$\alpha_0 = \omega'(t) = \frac{T}{mr} \qquad \omega(t) = \omega(t_i) + \frac{T}{mr}(t - t_i)$$

Non-uniform Circular Motion

Analogy to linear motion

Constant (angular) acceleration α_0 yields:

linearly increasing (angular) velocity $\omega(t)$;

quadratically increasing (angular) position

$$\theta(t) = \theta(t_\mathrm{i}) + \omega(t_\mathrm{i})(t - t_\mathrm{i}) + \tfrac{1}{2}\alpha_0(t - t_\mathrm{i})^2$$

Now change the winding radius on the spool

Mass m is at r, tension force applies at R

How does this change things ?

Archimedes' (lever arm) principle

$$a_t = v'(t) = \frac{R}{r}\frac{T}{m} \quad \Rightarrow \quad \alpha_0 = \omega'(t) = \frac{R}{r^2}\frac{T}{m}$$

Re-write as:

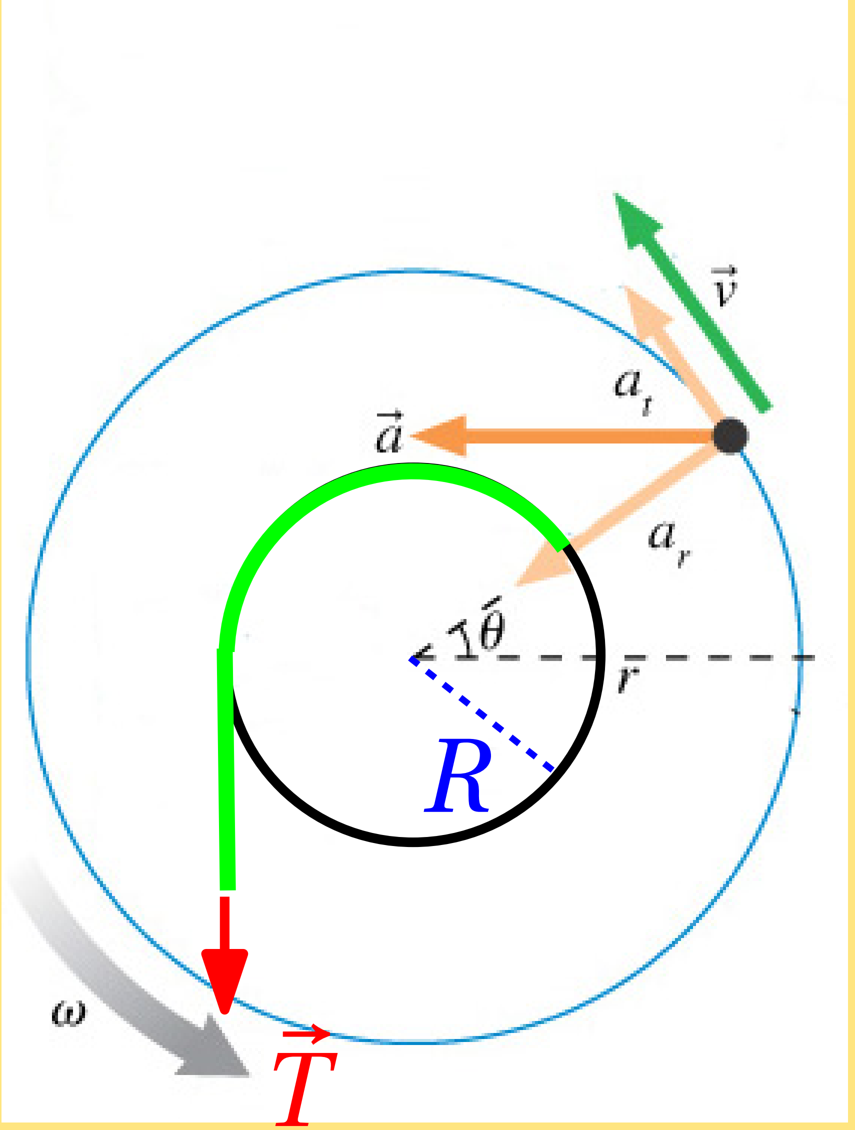
$$mr^2\alpha = RT$$

Looks like Newton's 2nd law ?

Rotational inertia: $I = mr^2$ mass times distance squared

Torque magnitude: $N = RF$ arm length times force

Works for rotation about fixed axis only !



Summary:

$$I\,\theta''(t) = N$$

analog of $mx''(t) = F_x$

For rigid bodies:

$$I = \gamma m r^2$$

γ geometrical factor

r characteristic length

Superposition in Motion

Projectile motion without drag

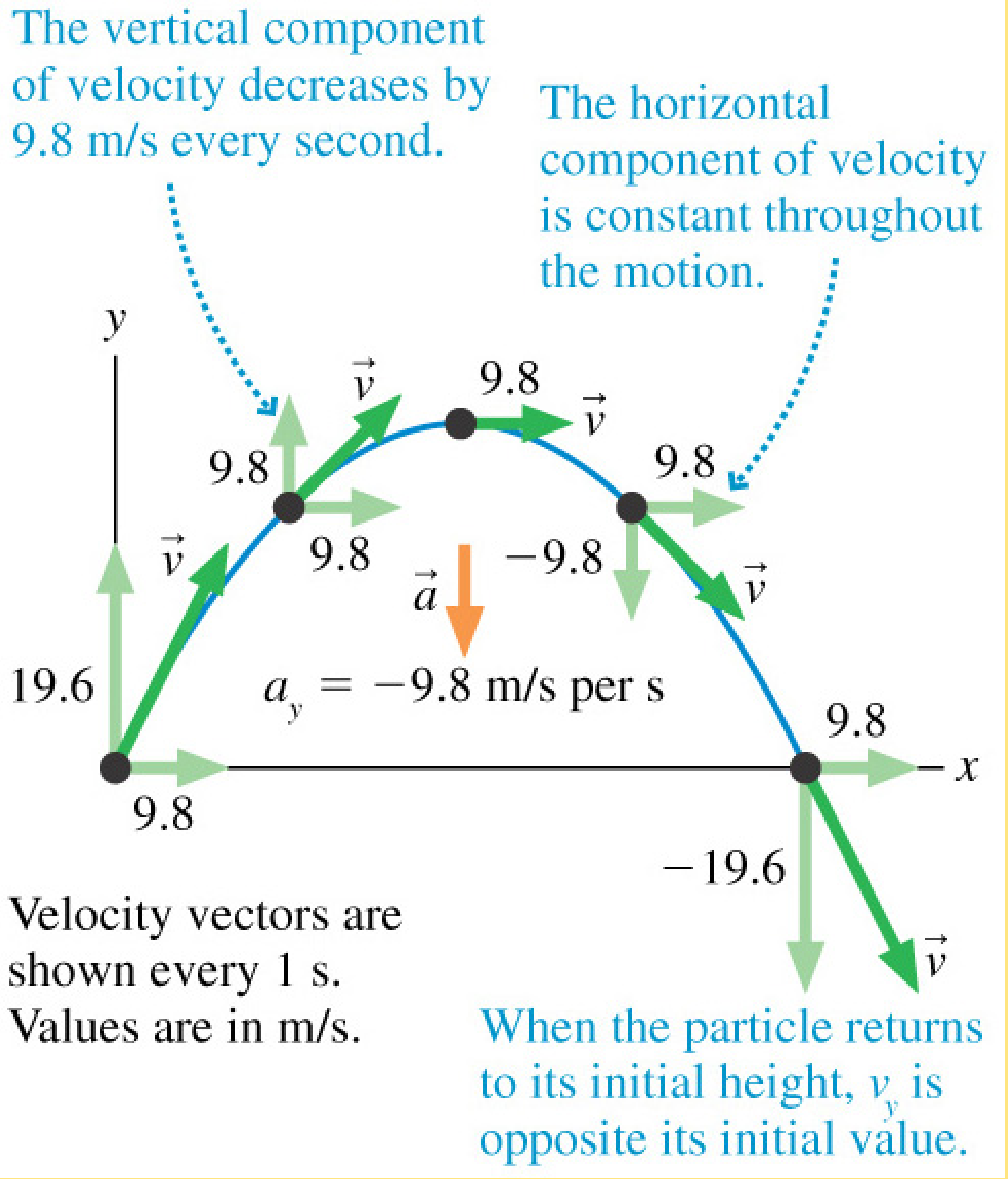
Shoot cannonball with $\vec{v}_\mathrm{i}$ at an angle

$\vec{v}_\mathrm{i} = (v_\mathrm{i}, \theta)$; $v_{\mathrm{i},x} = ?$; $v_{\mathrm{i},y} = ?$

$v_{\mathrm{i},x} = v_\mathrm{i} \cos\theta$; $v_{\mathrm{i},y} = v_\mathrm{i} \sin\theta$

$\vec{v}_\mathrm{i} = \hat{\imath} v_\mathrm{i} \cos\theta + \hat{\jmath} v_\mathrm{i} \sin\theta$

At all times: $\vec{v}(t) = \hat{\imath} v_x(t) + \hat{\jmath} v_y(t)$



Newton's 2^{nd} (component form):

$a_x(t) = F_x / m$; $a_y(t) = F_y / m$

$F_x = ?$; $F_y = ?$

$v'_x(t) = 0$; $v'_y(t) = -g$. $v_x(t) = ?$; $v_y(t) = ?$

$v_x(t) = v_{\mathrm{i},x}$; $v_y(t) = v_{\mathrm{i},y} - g t$. $[x(t), y(t)] = ?$

$x(t) = v_{\mathrm{i},x} t$; $y(t) = v_{\mathrm{i},y} t - \frac{1}{2} g t^2$. How do we show this is a parabola ?

Eliminating t in $y(t) = \ldots$ in favour of $x(t)$ does it. Find the time at which $y(t_\mathrm{f}) = 0$:

$t_\mathrm{f} = 2 v_{\mathrm{i},y} / g$. What is the horizontal displacement at this time ?

$x(t_\mathrm{f}) = \frac{2 v_{\mathrm{i},x} v_{\mathrm{i},y}}{g} = \frac{2 v_\mathrm{i}}{g} \sin\theta \cos\theta$. For which angle θ is this maximized ?