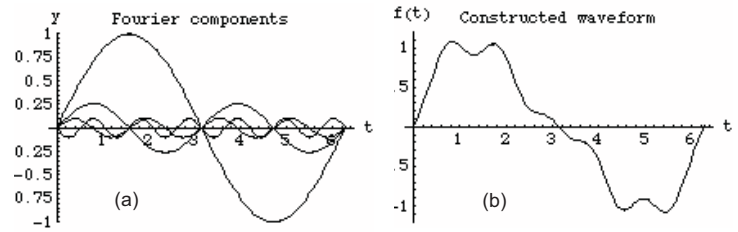# Lab 2: Fourier Analysis

Figure 1: (a) The wave with amplitude 1 oscillates at the fundamental frequency, and the other three waves are harmonics of the fundamental frequency. The Fourier components plotted are sin(t), 0.2sin(2t), -0.1sin(6t) and 0.1sin(4t). (b) The result of summing the Fourier components in (a).

1 Introduction

Refer to Appendix D for photos of the apparatus Joseph Fourier (1768-1830) was one of the French scientists during the time of Napoleon who raised French science to extraordinary heights. Working on the solution to a one-dimensional heat-diffusion equation, Fourier devised a method of expressing any periodic or non-periodic function as the sum of a series of sinusoidal terms. The principle of expressing an arbitrary function as the sum of a set of sinusoidal terms was revolutionary at the time and is called **Fourier theory** or **Fourier analysis**. Its applications extend far beyond the boundaries of heat-transmission theory, to any field involving waves.

In this experiment, you will construct different periodic waveforms by summing specific sinusoidal waves with predetermined amplitudes and phases on a computer. You will also investigate the concept of beats using an oscilloscope, by summing two sinusoidal waves (equal amplitudes) that have frequencies very close to each other. Finally, you will perform Fourier analysis of a sound wave generated by a metallic sound bar.

EXERCISES 1-15 PERTAIN TO THE EXPERIMENTAL SECTIONS.

2 Background

Let us first consider Fourier analysis for **periodic** functions. The Fourier theorem states that a periodic function can be constructed by summing a series of sinusoidal waves of different amplitudes, frequencies and phases. This series can possibly involve an infinite number of sine and/or cosine terms. One term in the series appears at the fundamental frequency, while the other terms oscillate at frequencies which are harmonics (i.e. multiples) of the fundamental frequency. A DC term at zero frequency is also included. Therefore, we can mathematically express a periodic function $f(t)$ in Fourier component form as follows,

$$f(t) = A_o + A_1 \cos(wt) + ... + B_1 \sin(wt) + B_2 \sin(2wt) + ... \quad (1)$$

where A_o is the DC component of the periodic function, $A_1 \cos(wt) + B_1 \sin(wt)$ is the fundamental component of the function, $A_n \cos(nwt) + B_n \sin(nwt)$ is the nth harmonic component of the function and $A_n \& B_n$ are known as the Fourier coefficients or amplitudes. The fundamental angular frequency is $w = \frac{2\pi}{T}$ where T is the period.

Using more compact notation, equation 1 can be rewritten as,

$$f(t) = A_o + \Sigma_{n=1}^{\infty} A_n \cos(nwt) + \Sigma_{n=1}^{\infty} B_n \sin(nwt) \quad (2)$$

$$f(t) = \Sigma_{n=0}^{\infty} A_n \sin(nwt + \varphi_n) \quad (3)$$

where φ_n is a phase angle. (Recall that sine and cosine functions differ in phase by 90°).

The construction of some periodic waveforms is illustrated in Figures 1 and 2. Figure 1a shows plots of various sinusoidal terms with different amplitudes, frequencies and phases. Summing these waveforms results in the waveform of Figure 1b.

As a second example, Figure 2b illustrates the construction of a square wave from some sinusoidal components plotted in Figure 2a.

The fundamental component in Figure 2a is sin(t). This means that w=1 is the fundamental frequency, the phase angle is zero and the amplitude of the fundamental component is 1. Some of the harmonics are (1/3)sin(3t), (1/5)sin(5t) and (1/7)sin(7t). The am-
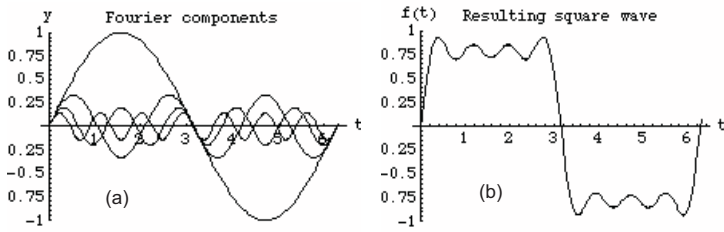
Figure 2: (a) The fundamental wave has amplitude 1, and the rest are harmonics of the fundamental. (b) Summing the waveforms in (a) results in an approximate square wave.

Figure 3: (a) The cosine function is an example of an even function with respect to the origin, and (b) sine is an odd function.

plitudes (or Fourier coefficients) of the various components can be derived mathematically, but the details are not presented here (see suggested readings).

Figure 2b shows a plot of the function,

$$f(t) = \sin(t) + \frac{1}{3}\sin(3t) + \frac{1}{5}\sin(5t) + \frac{1}{7}\sin(7t) \quad (4)$$

The waveform of figure 2b is only a crude approximation to a square wave. This is because we summed up only few terms in the series. If we sum up more terms of the series, a better approximation to the square wave is obtained. If we sum an infinite number of terms, we obtain the ideal square wave with perfectly sharp edges.

Notice that in equation 4, cosine terms are absent and only the odd harmonics of the fundamental frequency are involved. To understand why this is the case, recall the concept of **even and odd functions**. An even function (Figure 3a) is symmetric about the y-axis. This is expressed mathematically as f(-x) = f(x). On the other hand, an odd function (Figure 3b) is symmetric about the origin O of the rectangular coordinate system, and satisfies f(-x) = -f(x). A waveform is said to possess **half-wave symmetry** whenever the values in the first half of the period are just the negatives of the values in the 2nd half of the period or vice versa. The components required to construct a waveform with half-wave symmetry must be odd. Therefore, cosine terms are not necessary.

A square wave that is odd about the origin requires only sine terms (see equation 4). In addition, only odd harmonics of the fundamental frequency will appear in the Fourier series because the square wave is also even about $t = \frac{\pi}{2}$, hence the Fourier components must not vanish at $t = \frac{\pi}{2}$, and the sine of odd frequencies is appropriate, as illustrated in Figure 4.
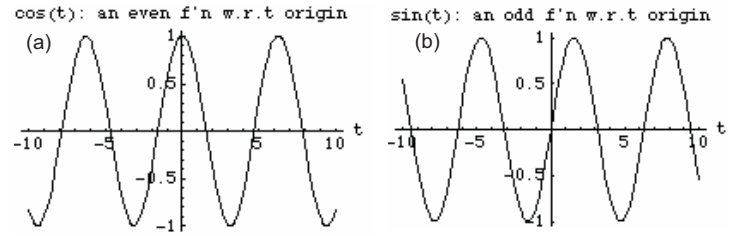
In the first experiment, you will study the construc-

tion of a square wave and some other periodic waveforms using a computer.

So far, we have defined waveforms in the time domain, i.e. as a function of time f(t). Fourier methods can also be developed in the frequency domain, i.e. F(w), which is called the **Fourier transform** of f(t). The Fourier transform gives the frequencies of the harmonics and their respective coefficients or amplitudes.

A plot of the Fourier coefficients (or amplitudes) A_n and/or B_n vs frequency is often very informative, since it shows at a glance the frequency components, the magnitude and the sign of the amplitudes. This type of plot is often referred to as the **frequency spectrum** or **frequency domain plot**. A sample spectrum is shown in Figure 5.

Fourier analysis is not limited to periodic waveforms. It can be extended to a waveform which does not repeat. In contrast to the construction of a periodic waveform where discrete frequencies were required, we need a continuous range of frequencies to construct a single waveform. The frequency spectrum will be a continuous function and the Fourier series of equation 1 is not useful in this case. Instead, the continuous Fourier transform is commonly used. Fourier analysis of non-periodic waveforms is not studied in this experiment. Refer to the suggested reading to learn more on continuous Fourier transforms.

Beats: When two waves with the same amplitude and having similar frequencies interfere, the amplitude of the resulting waveform varies sinusoidally. These variations of amplitude are called "beats". To show this mathematically, consider the superposition of the following two waves: $s_1 = a\cos(w_1 t)$ & $s_2 = a\cos(w_2 t)$ where a is the amplitude of each wave and w_1 and w_2 are the angular frequencies ($w = 2\pi f$).

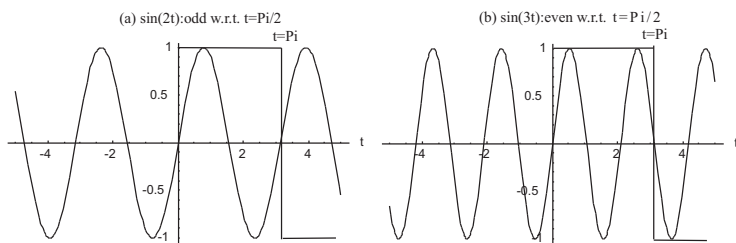If we assume that w_1 and w_2 have similar values,

Figure 4: (a) Even harmonics of the fundamental frequency are not useful in constructing the square wave since the even harmonics are not even w.r.t. $\frac{\pi}{2}$. (b) odd harmonics of the fundamental frequency are used since they are even about $\frac{\pi}{2}$.

$$s = s_1 + s_2 = 2a \cos(\frac{w_1 - w_2}{2}t) \cos(\frac{w_1 + w_2}{2}t) \quad (5)$$

Let $w' = \frac{w_1 - w_2}{2}$ be the difference angular frequency and $w = \frac{w_1 + w_2}{2}$ be the sum angular frequency. Then equation 5 becomes,

$$s = 2a \cos(w't) \cos(wt) \quad (6)$$

Equation 6 is a cosine function with angular frequency w. The term $2a \cos(w't)$ is the amplitude of the function. Notice the amplitude is oscillatory with the difference angular frequency w'.

The angular beat frequency is twice w' (refer to suggested reading), and is the difference between the interfering angular frequencies,

$$w_{beat} = w_1 - w_2 \quad (7)$$

and since $w = 2\pi f$, the beat frequency is,

$$f_{beat} = f_1 - f_2 \quad (8)$$

The high frequency of the waveform is just given by,

$$f = \frac{f_1 + f_2}{2} \quad (9)$$

Figure 6a shows two signals with close frequencies, which when interfere result in beats as shown in Figure 6b.
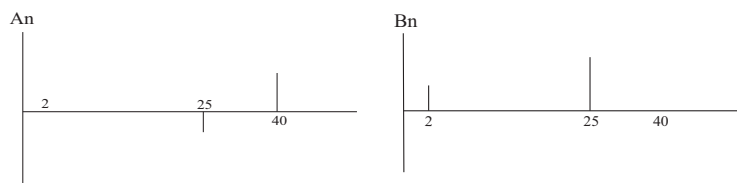

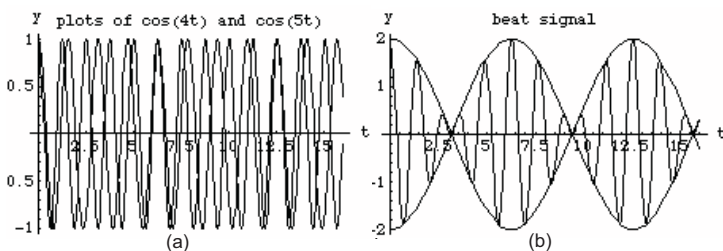Figure 5: Frequency spectrum for an arbitrary function

Figure 6: (a) plots of interfering waves (b) The angular beat frequency is (5-4)=1 rad/s, while the high (angular) frequency of the wave inside the envelope is (5+4)/2=4.5 rad/s.

3 Suggested Reading

D. Halliday, R. Resnick and J. Walker, **Fundamentals of Physics** (4th Edition)

E. Hecht and A. Zajac, **Optics** (especially p.323)

4 Apparatus

- Fourier Analysis software
- PASCO Fourier Synthesizer
- Oscilloscope
- Speaker
- Banana/BNC connectors for synthesizer, oscilloscope and speaker
- Sound bar and hammer
- Computer interface and digital oscilloscope software

5 Experiment I: Construction of periodic waveforms using Fourier analysis software

In this experiment, you will observe how some periodic waveforms are constructed. To construct a wave-

form, we need to know the amplitude of each of the components in the Fourier series of equation 1. It is possible to determine the amplitudes mathematically. However, you will be using computer software to find out the amplitudes and the harmonic components required for constructing the following waveforms,

1. Square wave

2. Triangle wave

3. Saw Tooth wave

If the Fourier analysis software is not already setup, open it by following these steps: win → main → file manager → fourier → fourier3.exe

Select **Square wave** from the menu. Record the values for the **harmonics** and **amplitudes** for the first 19 harmonics. You can press pause (P) to freeze the screen - it will make it easier to record the values.

Exercise 1: Sketch the waveform obtained by summing the first 19 harmonics. (For the square wave, notice the overshoot associated with the abrupt rising or falling edges).

Repeat for a,

- **Triangle wave** (choose triangle option from menu)

- **Saw tooth wave** (choose saw tooth from menu)

Exercise 2: Sketch the waveform obtained by summing the first 19 harmonics for the triangle and the saw tooth waves.

Exercise 3: Choose the following waveforms and only sketch them when they become identifiable:

- Half wave rectified sine wave

- Full wave rectified sine wave

- Sinusoidal burst (The sinusoidal burst signal is used in color TVs to keep track of the color on the set; it is called a color burst signal)

Exercise 4: For the waveforms used in Exercise 3, examine whether even only, odd only, or both even and odd harmonics are required for constructing these waveforms. (You don't need to record any of the harmonics or the amplitudes).

Exercise 5: Organize the 6 waveforms you studied into the following categories: those that consist of odd harmonics only, those that consist of even harmonics only, and those that require both even and odd harmonics. Explain why each waveform belongs to that particular category.

Consider the square wave you constructed in exercise 1, and recall the overshoot at the rising and falling edges of the wave. This overshoot is called the **Gibbs phenomenon**. It is due to the fact that only a finite number of Fourier terms are being used in the synthesis of the waveform. To resolve the Gibbs phenomenon, computer algorithms are devised which calculate variations of the Fourier coefficients. Using these modified coefficients, the overshoot at the edges of the square wave can be reduced.

Exercise 6a: Select the Hamming window, which is an algorithm to compute the modified Fourier coefficients. You have to indicate the number of the highest harmonic: choose 19. Record the harmonics and amplitudes for the Square wave only. Notice how the amplitudes differ from those in exercise 1.

Exercise 6b: Sketch the square wave obtained after summing 19 harmonics.

6 Experiment II: Synthesis of some periodic waveforms using a synthesizer

In experiment 2, you will use the amplitudes and the harmonics recorded for some of the waveforms in experiment 1, in order to synthesize those waveforms using the PASCO Fourier Synthesizer, and view them on the oscilloscope.

Description of the synthesizer: The front panel is arranged in 10 columns, each one controls the synthesizer's 10 channels. The two left columns labeled (1,1) produce 440Hz signals. These signals represent the fundamental frequencies. Each of the other columns 2, 3, 4, etc. represents the respective harmonic of the fundamental. For example, the 7th harmonic cor-

responds to column number 7, and has a frequency of 7x440Hz=3080Hz. The slide switches below the columns labeled (1) select one of the three waveforms: sine, triangle, and square. The next two slide switches are labeled $0 - 90°$ and $0 - 180°$; they control the phase, and a variable knob that can be used to select any phase between $0 - 360°$. If both switches and the variable knob are placed in the "0" position, then all the signals are in phase. The next row of variable knobs (below the phase knobs) is used to vary the amplitudes of the channel signal (clockwise increases the signal). Each channel has a red banana jack which can be used to monitor its waveform on the oscilloscope. Each channel can be connected/disconnected by sliding the in-out slide switch. Finally, the TRIGGER OUTPUT can be connected to the external trigger on an oscilloscope. When the TRIGGER OUTPUT is connected to an oscilloscope, the synthesizer produces a square wave signal at the fundamental frequency. This square wave is used to trigger (or start) the oscilloscope waveform on the left side of the screen. The 10K OUTPUT is used for the synthesized (or summed) waveform and the 8Ω OUTPUT is used for a speaker. Pressing the RESET button will reset the digital circuitry; it is required, if for instance, occasional transient signals shift the phases.

To get a feel for the instrument, connect the speaker to the 8Ω OUTPUT on the synthesizer and slide all the in-out switches to the **out** positions. Also, place all the phase switches and the knob to "0". Next, place all the amplitude knobs to max, i.e. fully clockwise. Place the summing amplifier gain knob in the "10 o'clock" position. Connect the oscilloscope to the 10K OUTPUT of the synthesizer. Trigger the oscilloscope externally by the TRIGGER OUTPUT of the synthesizer. Slide the in-out switch of column (1) on the left to the **in** position. Adjust the trigger level on the oscilloscope and the position of the signal. Use AC coupling on the oscilloscope, since we are interested in the AC component of the signal.

Exercise 7a: From the trace on the scope, determine the frequency ($f = \frac{1}{T}$ where T is the period) and the amplitude of the signal.

Exercise 7b: Place the first channel (1) back to the **out** position and repeat exercise 7a for the rest of the channels. Remember to switch **in** one channel at a time.

Exercise 8: Are all the harmonics correctly cali-

brated? Test this by taking the ratio of each harmonic to the fundamental frequency. The ratios should be: 1:2, 1:3, 1:4, etc.

Now using the data you recorded in experiment 1, synthesize the square wave. Use the second column (1) from the left on the synthesizer as your fundamental frequency. In order to use your amplitudes from experiment 1, multiply each by a factor of 10. For example, the amplitude 1 will now be represented by 10, etc. Adjust the amplitude values for the fundamental and the following harmonic and slide both switches to the **in** position.

Exercise 9: Does the output begin to look like a square wave?

Next, play with the phase. Switch the phase switch on the 3rd harmonic to the 180° position.

Exercise 10: Does it now begin to take the shape of a square wave? You can use the two channels of the oscilloscope to view separately the fundamental and the 3rd harmonic. Switch the 180° slide to 0° and then back to 180° position. Describe what happens to the signals when the phase is 0° and 180°.

Reconnect channel 1 of the oscilloscope to the 10K OUTPUT of the synthesizer. Continue adding the remaining harmonics to form your square wave. Remember you may have to adjust the phase to get the best square wave.

Exercise 11: Observe how the waveform changes as harmonics are added. Sketch what the "final" waveform looks like with the limited number of harmonics used.

Exercise 12: Synthesize the triangle and the saw tooth waveforms using the data recorded in experiment 1 and the method described above. Place all the in-out switches to the **out** position and reset all the phases to zero before starting to construct a new waveform.

7 Experiment III: Beats

In this part of the experiment, you will investigate beats, which are sinusoidal variations of the amplitude of a wave constructed by adding two waves of equal
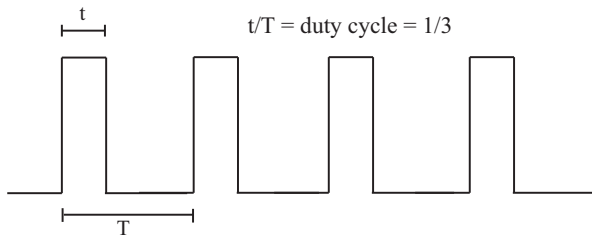
t/T = duty cycle = 1/3

Figure 7: Pulse waveform

amplitude and similar frequencies.

Connect the oscilloscope to the 10K OUTPUT on the synthesizer. Trigger the oscilloscope as in experiment 2. Switch in the 4th harmonic and adjust the amplitude to a convenient value. Switch out the 4th harmonic and switch in the 5th harmonic. Adjust the amplitude until it equals the amplitude of the 4th harmonic. Now switch in both the 4th and the 5th harmonics.

Exercise 13a: Sketch the observed beat signal.

Exercise 13b: Determine the low (beat) frequency of the "envelope" and the high frequency of the wave within the envelope. To measure the frequencies accurately, you may have to expand (for high frequency) or compress (for low frequency) the time scale on the oscilloscope.

Exercise 13c: Compare the values you obtained in exercise 13b with the theoretical predictions of equations 8 and 9.

Exercise 13d: Measure the amplitude of the envelope, and compare it with the prediction of equation 6.

Exercise 13e: Write the equation for the beat signal.

Exercise 13f: Discuss your results.

8 Experiment IV: Discrete Fourier Transform of a square wave

In this experiment, we obtain a discrete Fourier transform of a particular square wave by using the Fourier analysis program. Choose M from the main menu to construct the following waveform as shown in Figure 7.

To draw your plot on the grid on the computer screen, use the following commands: PgUp → cursor right (small right arrow on keyboard) → PgDn → ...

When you are done, press End → D to get the discrete Fourier transform, which is a graph of amplitudes vs. the harmonics.

Exercise 14a: Draw a plot of the frequency spectrum. Record the amplitudes and harmonics which synthesize this waveform.

Exercise 14b: Write down the equation which synthesizes this waveform for the first few terms.

9 Experiment V: Fourier analysis of a sound wave

In this experiment, you will use a sound bar as a source and a computer interface to input a sound wave into the computer and thus decompose it into its Fourier components. The computer software for this experiment functions as a digital oscilloscope; it can obtain a Fourier transform for a signal.

To start the program, exit Windows and return to DOS. At the C:¿ prompt type: CD SOUND ¡enter¿. Then type SOUND ¡enter¿. Left-click the mouse button on COM2 to accept the communication port. Right-click (and continue to hold) the mouse button in the graph area to produce a menu, go down to DISPLAY and move the mouse to the right to get the submenu. Select **One Graph**. With the help of your partner, simultaneously strike the "metal flap" with the soft hammer on the sound box and have your partner press the START button on the screen. When the signal is recorded, press the STOP button. You may want to repeat the experiment until you get a reasonably good signal.

To find the frequency components of the signal, double click the left mouse button on the graph area. Change the "Type of Graph" from Data to FFT (Fourier Transform) by pressing the button with the Data. Press **Try It**. If it is satisfactory, press **OK**.

Provide detailed sketches of the recorded signal and the frequency spectrum.

Exercise 15a: What is the peak frequency and its

error of the signal from the spectrum? How does it compare to the value on the sound box?

Exercise 15b: Are there any other frequency components in the spectrum? If yes, explain any possible reason(s).

Your lab report should include:

Answers to exercises 1-15 with relevant data tables, graphs, figures and qualitative comments.

Refer to Appendix C for Maple worksheets.