PHYS4020

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1. Ampère's law has a problem which Maxwell managed to fix

1.1. **Introduction.** Ampèrian loops are curious things. They allow us to relate a line integral around the loop to a surface integral over an area bounded by the loop. What does bounding mean, however, is the tricky question. In the simplest case, a circle bounds the area of a disk. That is a very common example. We are, however, allowed to deform the area. This is tantamount to the creation of a soap bubble: just before the bubble separates, as one is blowing through the ring (our Ampèrian loop) the surface is deformed. It is the area integral over this surface that is still related to the line integral. We will swallow this as a mathematical fact. In (1) we have on the left a line integral over a closed contour C, and on the right we are integrating the vector field $\vec{J}$ that flows through area A, which is assumed to be bounded by C.

$$\oint_C \vec{B} \cdot d\vec{l} = \iint_A \vec{J} \cdot d\vec{a} \tag{1}$$

In the simplest examples of determining the strength of the magnetic field around a wire, i.e., determining the strength of the field as a function of cylindrical radius $B_\varphi(s)$ life is easy: the line integral leads for any value of radius s to

$$2\pi s B_\varphi(s) = \mu_0 I_{\rm encl} \ . \tag{2}$$

For radii s larger than the wire radius, i.e., $s > a$ the enclosed current is always the total current, and we do not need to evaluate the area integral.

A first extension of this application is the case where the circle is inside the wire of radius a, i.e., for $s < a$. The geometry is still simple, the cross sectional area of the circle of radius s is now permeated by a fraction of the total current. One has to make some assumptions about the current density distribution: for a so-called volume current, the current density is simply independent of s, and the above law (1) has to be handled carefully, but is straightforward:

$$2\pi s B_\varphi(s) = \mu_0 I_{\rm encl} = \mu_0 \pi s^2 \frac{I_0}{\pi a^2} \ . \tag{3}$$

We did not have to worry much about the surface integral, since the volume current density is obviously total current divided by the total wire cross section, and this gets multiplied by the permeated area. The justification for this simple answer comes from the independence of the current density on radius s (this would be different for an AC current which is likely to reside more in the surface parts of the wire) and the fact that the area vector and the current density vector both point along $\hat{z}$.

1.2. **Example: charging a capacitor with a constant-current power supply.** We now move to a more interesting case. A charging capacitor with a fixed voltage applied (via a resistor) has a complicated current as a function of time. We make life easier by applying an electronic device, called a constant-current power supply. Within limits (i.e., for a finite time) this supply will ramp the voltage up in such a way that the charging current is constant. It uses a feedback control loop to do this (Electronics course). For us it means that while it is working (until the maximum permitted voltage is reached) a constant current is charging the capacitor plates. Thus, we expect the magnetic field to have the behaviour given by (2). We assume the wires to the left plate and from the right plate to the power supply are straight and define the $\hat{z}$ direction.

We will now discover a major flaw with (1). In the simplest setting our capacitor will be made up of a thick cylindrical wire of radius a, which is cut in half to form a gap of distance d. This is a legitimate capacitor (even though the capacitance will be small), it has plate area πa^2, and surface charge is building up on the opposing faces of the wire where it is cut. An electric field $E_z(t)$ will build up as a result of the accumulating surface charges, which are positive on one side, and negative on the other. The potential difference $\Delta V(t) = E_z(t)d$ across the gap is provided by the power supply, and the wire is assumed to be ideal (negligible resistance $R \approx 0$).

All our Amperian loops in this section will be circles centered on the wire. We will, however, vary the surfaces: the simple disk area encircled by the loop, on the one hand, and a cup geometry on the other, where the rim of the cup will be the loop, and the surface area will be that of the bottom and of the sides (mantle) of the cup. There will also be a choice where to place the rim, whether around the current-carrying wire, or in the gap area of the capacitor. We will be looking for consistency in the magnetic field calculations, which are obtained at the position of the loop).

In Fig. 1 we show a scenario that points to a big problem with Amp̀re's law. The mathematically legitimate surface area deformation is chosen such that no part of the surface is permeated by current. The contradiction with the result (2) obtained before is obvious. Maxwell fixed this problem by postulating an additional term that is associated with the time-varying electric field inside the capacitor. In fact, it is the time derivative of the electric displacement field

$$\vec{J}_d \equiv \frac{\partial \vec{D}}{\partial t} = \epsilon \frac{\partial \vec{E}}{\partial t} \tag{4}$$

which is called displacement current which will restore sanity. The time-varying electric field in the gap region is associated with the build-up of surface charge $Q(t)$ on the disk areas of radius a at the wire ends. If current is carried into the capacitor from the left the left disk charges positively, while the opposing right side charges negatively (the blue lead is carrying positive charge to the right).

We can also ask what should happen to the magnetic field in the region surrounding the gap (and also inside the gap). Let us flip the cylindrical cup around to have the rim, i.e.,

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PL2 = Graphics3D[{EdgeForm[Directive[Thin, Dashed]], Lighter[Blue, 0.6],
   Cylinder[{{1.3, 0, 0}, {4, 0, 0}}, 1 / 2]}];
PL3 = ParametricPlot3D[{v, 0.75 Cos[u], 0.75 Sin[u]}, {u, 0, 2 π},
  {v, 0, 0.9}, PlotStyle → {Opacity[.2], Green}, Mesh → None];
PL4 = RegionPlot3D[y^2 + z^2 < 0.75^2, {x, 0.899, 0.9}, {y, -1, 1},
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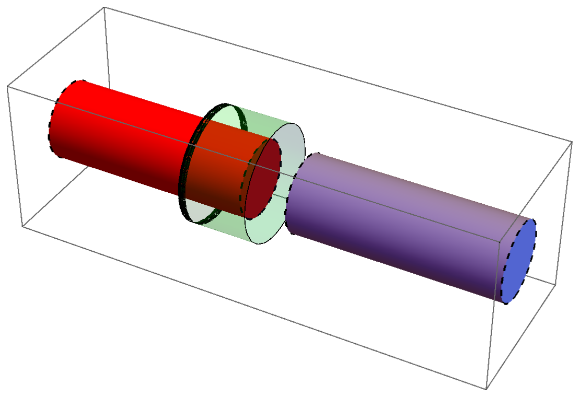

FIGURE 1. A wire (shown in red) from the left is carrying a constant current despite the gap in the middle. The wire continues to carry the current to the right (shown in blue). This is possible when the ends are connected to a power supply that ramps up the voltage linearly in time. An Ampèrian loop surrounds the wire to measure the magnetic field there. Its surface area is deformed into a cylindrical cup, shown in light green with the bottom of the cup in the gap area. This surface area is not permeated by electric current. Thus, the naïve answer would be that the magnetic field surrounding the wire at the location of the black loop should vanish. This contradicts the answer (2) obtained for the circular area surrounded by the Ampèrian loop.

our Ampèrian loop (AL) surrounding the gap. There is no reason why the magnetic field should have any discontinuities, i.e., we can expect $B_\varphi(s)$ to have the same strength for $s > a$ in the gap region, as in the region surrounding the current-carrying wire.

This expected result that there is no discontinuity in $B_\varphi(s)$ as a function of z, which means that there continues to be no z-dependence in the magnetic field is consistent with the calculation shown for the surface area displayed in Fig. 2. However, if one was not using the cup geometry, but simply asked what current was permeating the circular area enclosed by the AL of Fig. 2, then the naïve answer would be that there is no magnetic field surrounding the gap region. Again, the inconsistency with mathematics is resolved by considering the 'displacement current' (4). This postulated 'displacement current' does

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PL5 = ParametricPlot3D[{v, 0.75 Cos[u], 0.75 Sin[u]},
   {u, 0, 2 π}, {v, 0.95, 1.05}, PlotStyle → {Black}, Mesh → None];

PL4 = RegionPlot3D[y^2 + z^2 < 0.75^2, {x, 0, 0.01}, {y, -1, 1},
   {z, -1, 1}, PlotStyle → {Darker[Green, 0.3], Opacity[0.2]}, Mesh → None];

Show [PL3, PL5, PL4, PL1, PL2, PlotRange → {{-2, 4}, {-1, 1}, {-1, 1}}, Axes → None]
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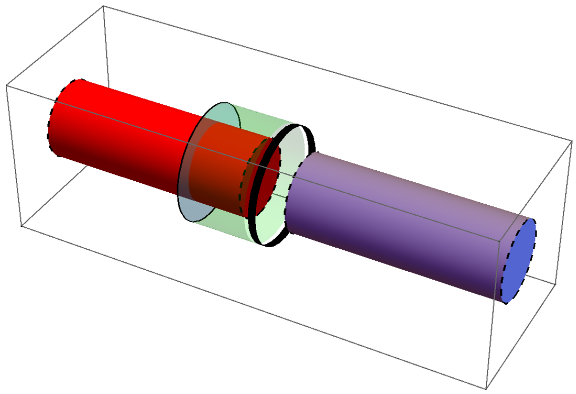

FIGURE 2. Compared to Fig. 1 the cup has been reversed to have its bottom permeated by the current in the left wire (shown in red), but the rim which is our Ampèrian loop is surrounding the gap region.

not contribute in the geometry of Fig. 2, since the direction is along $\hat{z}$, and it is confined to the gap region (realistically with a small amount of fringing).

1.3. **Derivation of the Ampère-Maxwell law.** A formal derivation of the Ampère-Maxwell law which includes the displacement current (4) is straightforward. Our example showed that there was something amiss when applying Ampère's law in its integral form. In general, starting from the differential form of the original law

$$\nabla \times \vec{B} = \mu_0 \vec{J} \tag{5}$$

one applies the divergence on both sides. The left-hand side vanishes by a mathematical theorem (the divergence of the curl of any vector field vanishes). Thus, we have

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_0 \nabla \cdot \vec{J} \ , \tag{6}$$

or

$$\nabla \cdot \vec{J} = 0 \ . \tag{7}$$

The question is: under what conditions is this true? After all, we used Ampère's law without problems before. The answer comes through the continuity equation:

$$\frac{\partial \rho(\mathbf{r}, t)}{\partial t} + \nabla \cdot \vec{J}(\mathbf{r}, t) = 0 \ . \tag{8}$$

Our example violates Ampère's law even though we have a steady current I_0 in the wire. Due to the gap in the wire there is a time-dependent accumulation of surface charge:

$$\sigma_s(t) = \frac{Q(t)}{\pi a^2} = \frac{I_0 t}{\pi a^2} \ , \tag{9}$$

where we assume that the current was turned on at time $t = 0$. We obtain charge $+Q(t)$ on one side of the gap, and $-Q(t)$ on the other. There is a homogeneous electric field in the capacitor gap

$$D_z(t) = \epsilon E_z(t) = \sigma_s(t) \ . \tag{10}$$

In a more general context one would have time-varying volume charge $\rho(\mathbf{r}, t)$ which in accord with Gauss' law is equivalent to the divergence of an electric field, for the general case of a dielectric medium (for vacuum set $\epsilon = \epsilon_0$)

$$\nabla \cdot \vec{D} = \rho \ . \tag{11}$$

Now take the time derivative to arrive at

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial \nabla \cdot \vec{D}}{\partial t} \ . \tag{12}$$

This implies that the previously used Ampère law cannot hold when there are time-dependent electric fields involved, since the divergence of the current does not vanish in this case, i.e., there is a conflict between (12) and (7).

Maxwell resolved the conflict by generalizing the law. We replace the current $\vec{J}$ in (1) or in (5) by the sum of the actual current and the displacement current expression (4). In differential form

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial \vec{D}}{\partial t} \ . \tag{13}$$

On account of (12) the divergence of the right-hand side of the new Ampère-Maxwell law vanishes!

1.4. **Example: charging a parallel-plate capacitor with a constant-current power supply.** We now step up the level of sophistication by turning to a parallel-plate capacitor, which will allow us to discuss situations where field fringing is negligible, and we will also obtain an understanding of how current flows to charge the plates, and what the consequences are for the magnetic field. We consider a wire of small radius a, thin disk-shaped capacitor plates of radius $R >> a$, and again use the trick of a constant-current power supply, which ramps up the voltage in such a way that the current in the wires is constant. The power supply ramps the applied voltage linearly, such that the voltage across the capacitor implies linear growth of total charge on the plates in accord with

$$\Delta V_C = \frac{Q(t)}{C} \ , \tag{14}$$

where the capacitance is given as in terms of plate area A and plate separation d as

$$C = \frac{\epsilon\, A}{d} \ . \tag{15}$$

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PL1 = Graphics3D[
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PL2 = Graphics3D[{EdgeForm[Directive[Thin, Dashed]],
   Lighter[Blue, 0.6], Cylinder[{{1.5, 0, 0}, {4, 0, 0}}, 1 / 20]}];
PL5 = ParametricPlot3D[{v, 1.2 Cos[u], 1.2 Sin[u]}, {u, 0, 2 π},
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PL6 = Graphics3D[{EdgeForm[Directive[Thick]], Red,
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Show [PL1, PL2, PL5, PL6, PL7, PlotRange → {{-2, 4}, {-2, 2}, {-2, 2}}, Axes → None]
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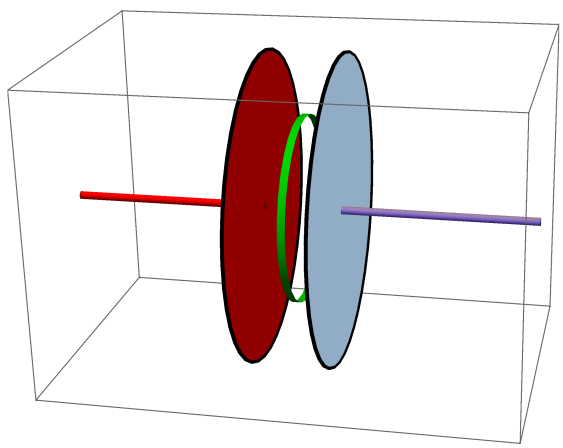

FIGURE 3. A parallel-plate capacitor fed by thin wires with an Ampèrian loop in the gap region. The surface area for now is the circular area enclosed by the loop. No free-charge current passes through this area, but the area is permeated by the increasing-with-time electric field between the plates.

For an air-filled capacitor $\epsilon = \epsilon_0$. The charge is assumed positive $+Q(t)$ on the left plate (shown in red), and negative $-Q(t)$ on the right plate (shown in blue).

The magnetic field $B_\varphi(s)$ in the gap region is calculated from the Ampère-Maxwell law by integrating the displacement current permeating the thin-disk area surrounded by the loop of radius s:

$$2\pi s B_\varphi(s) = \mu_0 \pi s^2 \frac{\partial D_z}{\partial t} \ . \tag{16}$$

Here we used the fact that the electric field $E_z(t) = D_z(t)$ is independent of s, and also of z, since it is homogeneous in between the plates, which is certainly true for $s < R$.

We learn that the magnetic field strength rises linearly with cylindrical radius s, and is constant in time (since the charging current is kept constant by the sophisticated power supply:

$$B_\varphi(s) = \frac{\mu_0\,\epsilon\,s}{2}\,\frac{dE_z}{dt} \quad \text{for} \quad a < s < R\ . \tag{17}$$

We replaced the partial derivative by a simple derivative since $E_z(t)$ does not depend on s, z, φ in the region of interest. To compare with the magnetic-filed result surrounding the wire we need to re-express the electric field strength in terms of the current.

First we use the known voltage-field-strength relation $E_z = \Delta V_C/d$, then we apply (14):

$$B_\varphi(s) = \frac{\mu_0\,\epsilon\,s}{2}\,\frac{d\Delta V_C}{dt}\,\frac{1}{d} = \frac{\mu_0\,\epsilon\,s}{2}\,\frac{dQ}{dt}\,\frac{1}{C\,d}\ . \tag{18}$$

Making use of the capacitor formula, and the definition of the current in terms of rate of change of charge with time, we find

$$B_\varphi(s) = \frac{\mu_0\,s}{2}\,I_0\frac{1}{A} = \frac{\mu_0 I_0 s}{2\pi R^2} \quad \text{for} \quad a < s < R\ , \tag{19}$$

where in the last step we expressed the plate area in terms of the plate radius R.

This result should be compared with the magnetic field surrounding the wire, which was derived in (2):

$$B_\varphi(s) = \frac{\mu_0 I_0}{2\pi s} \quad \text{for} \quad s > a\ , \tag{20}$$

Obviously, there is a difference! The field in the capacitor gap area grows linearly with s, while the field surrounding the wire gets weaker with distance from the wire. This leads to a discontinuity in the magnetic field strength. When the cylindrical radius approaches the plate radius $s \to R$, however, the two results do agree. For $s > R$ the result for the gap region can be extended, by taking into account that the electric field rapidly drops to zero, and that no more displacement current density is enclosed for $s > R$. Thus, for z values in the gap region, but radii $s > R$ the magnetic fields calculated around the wire, and around the capacitor region do agree.

1.5. **The magnetic field discontinuity.** What follows is a nice application of the magnetic field boundary condition associated with surface currents. The reason why $B_\varphi(s)$ is allowed to be discontinuous as we step from a z region outside the capacitor gap, but at $s < R$, say close to the wire (where the magnetic field is strong) into the capacitor gap, (where for small $s > a$ but $s < R$ the magnetic field is weak), is that there is a surface current radially outward on the capacitor plate that charges positively.

This radially directed surface current associated with how the total current I_0 deposits overall charge $Q(t) = I_0 t$ has a surface current density that varies with s: at $s = a$ there is a maximal surface current density, and as one approaches the plate radius $s = R$ the surface current density has to drop off, since some of the charge was deposited at smaller s. Surface current density K_f is given as current per unit length. In this case the length is the circumference $2\pi s$. This current density has a magnetic field associated with it, as explained, e.g., for an infinite sheet of current in Example 5.8 in Gr4e. The boundary condition is summarized in (5.76) in Gr4e:

$$\vec{B}_{\text{above}} = \vec{B}_{\text{below}} + \mu_0(\vec{K} \times \hat{n})\ , \tag{21}$$

where $\hat{n}$ is the normal area to the surface in which the current flows.

For our example $\hat{n} = \hat{z}$, the surface current density is $\vec{K} = K(s)\hat{s}$, and so the magnetic field produced by the surface current is in the $\hat{\varphi}$ direction, i.e., it is deemed responsible for the discontinuous behaviour in $B_\varphi(s)$ observed in the previous subsection (for $a < s < R$).

Our problem is now to figure out the functional dependence of the surface current density

$$K(s) = \frac{I(s)}{2\pi s} \ . \tag{22}$$

The total amount of current at radius s should satisfy the following boundary conditions:

$$I(a) = I_0 \quad \text{and} \quad I(R) = 0 \ . \tag{23}$$

Since current and deposited charge are proportional, we should expect a quadratic dependence on s, since at larger radius s there is more charge associated with the area. We can show this formally, by making use of the fact that the surface charge density

$$\sigma_f(t) = \frac{Q(t)}{\pi R^2} \tag{24}$$

does not depend on s. The total charge held by the part of the plate with radius s (we will assume $a \approx 0$ to simplify matters), can then be written as

$$Q_s(t) = \sigma_f(t)\pi s^2 = Q(t)\frac{s^2}{R^2} \ . \tag{25}$$

Thus, we find for the current $I_s(s) = dQ_s/dt$:

$$I_s(s) = I_0 \left(1 - \frac{s^2}{R^2}\right) \ . \tag{26}$$

For small $s > a \approx 0$ we obtain the full charging current, and as the plate radius is approached the current $I(s)$ drops rapidly since there is more surface area being charged at larger s.

Our expression for the surface current density then reads

$$K_s(s) = \frac{I_0}{2\pi s}\left(1 - \frac{s^2}{R^2}\right) \quad \text{for} \quad a < s < R \ . \tag{27}$$

Now go to the boundary condition (21). Rotate the set-up, such that the $\hat{z}$ axis is vertical, the current flows along $\hat{z}$ and charges the bottom plate. The current is radially outward. The cross product results in a contribution that is clockwise, i.e., $B_\varphi(s)$ above the plate (in the gap region) is reduced by the magnetic field caused by the surface current.

$$B_\varphi(s) = \frac{\mu_0 I_0}{2\pi s} - \mu_0 \frac{I_0}{2\pi s}\left(1 - \frac{s^2}{R^2}\right) = \frac{\mu_0 I_0 s}{2\pi R^2} \ . \tag{27}$$

The net result is exactly what was obtained from integrating the displacement current density (19).

At the top plate, which is charging negatively, the surface current density is inwards to represent the nominal flow of current. As a result the surface current causes a counter-clockwise magnetic field contribution, i.e., the weaker magnetic field between the plate is switched back to the expected result for a current-carrying wire.

1.6. **Consistency with the Ampère-Maxwell law (13).** The remaining task would be to apply the cup geometry and to verify that the justified result for the magnetic field discontinuity, obtained from the boundary condition (21) can also be explained by the Ampère-Maxwell law. One example would be to have the Ampèrian loop and surface area in analogy to Fig. 1. The cup is assumed to have a cylindrical radius $s < R$. In this case the field is calculated for the region outside the capacitor, but with a surface area which has two contributions: one from the cylinder mantle, where a radially outward free current density $K(s)$ crosses. Another contribution would be the displacement current density integrated over the bottom area of the cup. The two contributions will add up to provide the known simple result. The geometry is shown in Fig. 4.

Another choice would be the analog of Fig. **??** for $a < s < R$. Now the field is calculated in the capacitor gap region. The bottom of the cup is permeated by the full current carried by the wire. Counting this as going into the surface we realize that the surface current density $K(s)$ on the cylinder mantle is in the opposite direction. This is what causes the reduced values of $B_\varphi(s)$ inside the gap region. We leave these calculations as exercises, where the difficult steps (such as calculating $K(s)$ have been taken care of in the previous subsections.

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PL4 = RegionPlot3D[y^2 + z^2 < 1.2^2, {x, 0.899, 0.9}, {y, -2, 2},
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PL5 = ParametricPlot3D[{v, 1.2 Cos[u], 1.2 Sin[u]}, {u, 0, 2 π},
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PL6 = Graphics3D[{EdgeForm[Directive[Thick, Red]],
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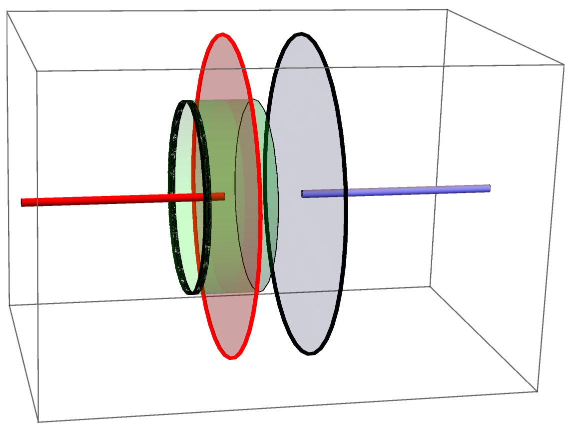

Figure 4. A parallel-plate capacitor fed by thin wires with an Ampèrian loop around the wire feeding the charge current for the left plate. The surface area is now a cup whose rim is the Ampèrian loop. The current-carrying wire does not cross the surface area. No free-charge current passes through the bottom of the cup, but this area is permeated by the increasing-with-time electric field between the plates. The cylinder mantle side area of the cup at radius s is permeated by the plate-charging surface current $I(s)$, which has surface current density $K(s)$ given by (27) associated with it. In this case the strong magnetic field surrounding the wire at radius s is caused by a combination of free current $I(s)$ and displacement current through the bottom of the cup.