arge Z_1e and a nucleus with charge Z_2e, as in Rutherford scattering. However, Coulomb potential does not vanish fast enough for large r to ensure that (13.7) an asymptotic solution to the Schrödinger equation in this case, and thus our malism is not appropriate for a pure Coulomb interaction. Nonetheless, we n consider the factor $e^{-m_0 r}$ as a mathematically convenient way to introduce screening that actually occurs in Rutherford scattering, where the electrons thin the atom shield the incident α particle from the nucleus until the α particle netrates the electron cloud. Recall that the size of the atom is on the order of angstrom, while the size of the nucleus is between 10^4 and 10^5 times smaller.

From (13.45)

$$f(\theta, \phi) = -\frac{\mu g}{2\pi\hbar^2}\int d^3r'\,\frac{e^{-m_0 r'}}{r'}e^{i\mathbf{q}\cdot\mathbf{r}'} \tag{13.51}$$

order to carry out the integrals, it is convenient to choose our dummy integration riables so that the z' axis is parallel to $\mathbf{q}$. Then $\mathbf{q}\cdot\mathbf{r}' = qz' = qr'\cos\theta'$, where is the usual polar angle in spherical coordinates. Thus

$$f = -\frac{\mu g}{2\pi\hbar^2}\int_0^\infty dr'\int_0^{2\pi} d\phi'\int_0^\pi d\theta'\sin\theta' r' e^{-m_0 r'}e^{iqr'\cos\theta'} \tag{13.52}$$

rrying out the angular integrals first, we obtain

$$\begin{aligned} f &= \frac{\mu g i}{\hbar^2 q}\int_0^\infty dr'\,e^{-m_0 r'}\left(e^{iqr'} - e^{-iqr'}\right) \\ &= \frac{-2\mu g}{\hbar^2(m_0^2 + q^2)} \end{aligned} \tag{13.53}$$

ote that

$$\begin{aligned} q^2 &= (\mathbf{k}_i - \mathbf{k}_f)^2 \\ &= (k^2 - 2\mathbf{k}_i\cdot\mathbf{k}_f + k^2) \\ &= 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2} \end{aligned} \tag{13.54}$$

d therefore $f = f(\theta)$ only. The angle θ is shown in Fig. 13.7. Thus

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \frac{4\mu^2 g^2}{\hbar^4[m_0^2 + 4k^2\sin^2(\theta/2)]^2} \tag{13.55}$$

he differential cross section depends only on the angle θ and not on the angle because of azimuthal symmetry under rotations about the z axis for a spherical otential.

We now specialize to the case of Coulomb scattering. At energies and/or gles such that $4k^2\sin^2(\theta/2) \gg m_0^2$, the expression (13.55) for the differential oss section reduces to

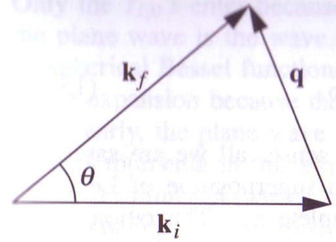


FIGURE 13.7
The incident wave vector $\mathbf{k}_i$, the scattered wave vector $\mathbf{k}_f$, and the vector $\mathbf{q}$.

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{\mu^2(Z_1Z_2e^2)^2}{\hbar^4 4k^4\sin^4(\theta/2)} \\ &= \frac{(Z_1Z_2e^2)^2}{16E^2\sin^4(\theta/2)} \end{aligned} \tag{13.56}$$

the famous result for Rutherford scattering. Interestingly, the dependence on Planck's constant has disappeared entirely. This differential cross section (13.56) is in complete agreement with that obtained from a classical analysis of Coulomb scattering, as well as with that obtained from an exact solution using quantum mechanics. Without this rather fortunate agreement between the classical and quantum results, the historical development of quantum mechanics would almost certainly have been quite different. It was the agreement between (13.56) and experiment that led Rutherford, *before* the advent of quantum mechanics, to the nuclear model of the atom. In classical physics, this model could not be stable as the electrons in their classical orbits radiated away their energy and spiraled into the nucleus. Bohr first addressed these problems with his own model, with its stationary states and discrete energies, in the crucial transition between classical and quantum mechanics.

13.4 THE PARTIAL WAVE EXPANSION

In general, the Born approximation is a high-energy approximation for calculating the differential cross section. There is another approach, known as the partial wave expansion, that is most useful at low energies and is therefore somewhat complementary to the Born approximation.

As we have seen in Rutherford scattering, if the potential energy is spherically symmetric, the scattering amplitude is a function of θ only:

$$f(\theta, \phi) = f(\theta) \tag{13.57}$$

We begin by writing

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l(k)P_l(\cos\theta) \tag{13.58}$$

a superposition of the partial waves, where

$$P_l(\cos\theta) = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l,0} \tag{13.59}$$

a Legendre polynomial (see Problem 9.17). In a sense, all we are saying in 3.58) is that we can write any function of θ as a superposition of Legendre olynomials. After all, these functions form a complete set. The rationale for xpressing the coefficients in the expansion in the form (13.58) will become clear hortly. For now, let us note that the value of the coefficient $a_l(k)$ will, in general, epend on the value of the energy, a dependence exhibited explicitly by indicating at the partial wave is a function of k.

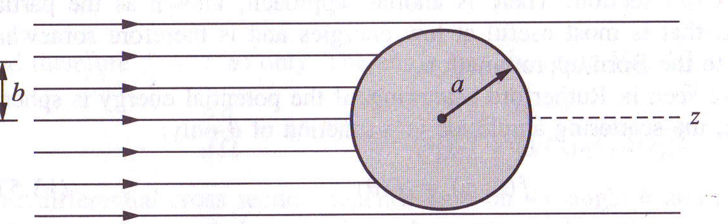
Although in principle we have traded $f(\theta)$ for an infinite set of partial waves, e utility of (13.58) comes from the fact that at low energies only a few of the (k) are significantly different from zero. To see why, we first give a heuristic emiclassical argument. In general, a beam of particles incident on a target in scattering experiment with a broad spectrum of impact parameters consists of any different orbital angular momenta, as can be seen by evaluating $\mathbf{r}\times\mathbf{p}$ for ach impact parameter (see Fig. 13.8). However, if the potential energy has a nite range a, scattering will occur only for those impact parameters that are less an a. Thus for interaction there is a maximum angular momentum, whose value roughly given by $\hbar l_{\max} = ap = a(\hbar k)$, or $l_{\max} = ak$. The lower the energy nd the smaller the value of k, the fewer angular momentum states can interact ith the target.

To start our partial wave analysis in quantum mechanics, we express the cident plane wave in the form (see Problem 13.8)

$$e^{ikz} = e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l(2l+1) j_l(kr) P_l(\cos\theta) \tag{13.60}$$

hich can be considered as a special case of the more general expansion

$$\psi(\mathbf{r}) = \sum_l c_l R_l(r) Y_{l,0}(\theta) \tag{13.61}$$



IGURE 13.8
classical representation of the incident flux in terms of particles following well-defined trajectories. hose particles with impact parameter b possess orbital angular momentum $|\mathbf{L}| = |\mathbf{r}\times\mathbf{p}| = b\hbar k$. nly particles with impact parameters less than or equal to the range a of the potential energy would teract with the target.

Only the $Y_{l,0}$'s enter because the plane wave (13.60) is independent of ϕ. Since the plane wave is the wave function of a free particle, the radial functions must be spherical Bessel functions. We must discard the spherical Neumann functions in the expansion because they blow up at the origin, as we discussed in Chapter 10. Clearly, the plane wave is finite at the origin. Note that the appearance of all angular momenta in this expansion is consistent with a picture of a plane wave, which is infinite in extent, as having all impact parameters.

What can we say about the asymptotic form of the full wave function (13.7), including the scattered wave, in terms of partial waves? Since we are searching for a solution of the Schrödinger equation for $r \to \infty$, a region in which $V = 0$ but one that excludes the origin, we must include both spherical Bessel and Neumann functions in our general expression for the function $R_l(r)$ in (13.61):

$$\psi(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \sum_l [A_l j_l(kr) + B_l \eta_l(kr)]\, P_l(\cos\theta) \tag{13.62}$$

The asymptotic expressions for the Bessel functions themselves are given by

$$j_l(kr) \underset{r\to\infty}{\longrightarrow} \frac{\sin(kr - l\pi/2)}{kr} \qquad \eta_l(kr) \underset{r\to\infty}{\longrightarrow} -\frac{\cos(kr - l\pi/2)}{kr} \tag{13.63}$$

Substituting these forms into (13.60) and (13.62), we obtain

$$e^{ikz} \underset{r\to\infty}{\longrightarrow} \sum_{l=0}^{\infty} i^l(2l+1)\frac{\sin(kr - l\pi/2)}{kr} P_l(\cos\theta) \tag{13.64}$$

for the incident wave and

$$\begin{aligned}\psi(\mathbf{r}) &\underset{r\to\infty}{\longrightarrow} \sum_l \left[A_l \frac{\sin(kr - l\pi/2)}{kr} - B_l \frac{\cos(kr - l\pi/2)}{kr}\right] P_l(\cos\theta)\\ &= \sum_{l=0}^{\infty} C_l \frac{\sin[kr - l\pi/2 + \delta_l(k)]}{kr} P_l(\cos\theta)\end{aligned} \tag{13.65}$$

for the complete wave function, where in the last step we have combined the sine and cosine into a sine function with its phase shifted by $\delta_l(k)$. Comparing (13.64) and (13.65), we see that the effect of the potential is to introduce a phase shift in the asymptotic wave function. Figure 13.9 shows qualitatively how this happens.

We can express (13.65) in the form

$$\psi \underset{r\to\infty}{\longrightarrow} \sum_{l=0}^{\infty} C_l \frac{e^{i(kr - l\pi/2 + \delta_l)} - e^{-i(kr - l\pi/2 + \delta_l)}}{2ikr} P_l(\cos\theta) \tag{13.66}$$

which contains both incoming and outgoing spherical waves. What is the source of these incoming spherical waves? They must be due to the presence of the incident plane wave in the full asymptotic wave function. If we rewrite (13.64) as

$$e^{ikz} \underset{r\to\infty}{\longrightarrow} \sum_{l=0}^{\infty} i^l(2l+1)\frac{e^{i(kr - l\pi/2)} - e^{-i(kr - l\pi/2)}}{2ikr} P_l(\cos\theta) \tag{13.67}$$

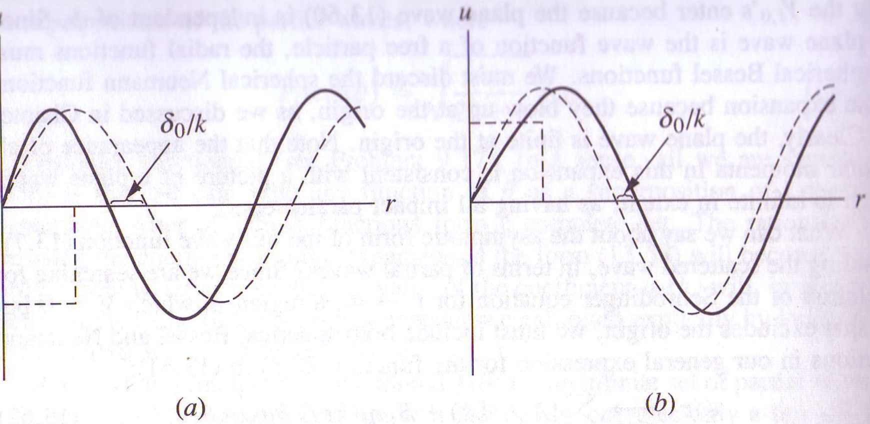


FIGURE 13.9
A depiction of how the potential energy affects the phase of a wave. (*a*) A potential well (an attractive potential) produces a positive phase shift ($\delta_0 > 0$) for the radial function $u = rR$ while in (*b*) a potential barrier (a repulsive potential) generates a negative phase shift ($\delta_0 < 0$). The dashed curve shows u when $V = 0$ in each case.

we see these incoming spherical waves explicitly. Since[3]

$$\psi - e^{ikz} \underset{r\to\infty}{\longrightarrow} f(\theta)\frac{e^{ikr}}{r} \tag{13.68}$$

which is an outgoing spherical wave only, the incoming spherical waves must cancel if we subtract (13.67) from (13.66), which implies that

$$C_l = (2l+1)e^{il\pi/2}e^{i\delta_l} \tag{13.69}$$

With this result, we see that

$$\begin{aligned} f(\theta) &= \sum_{l=0}^{\infty}(2l+1)\frac{1}{2ik}(e^{2i\delta_l}-1)P_l(\cos\theta) \\ &= \sum_{l=0}^{\infty}(2l+1)\frac{e^{i\delta_l}}{k}\sin\delta_l P_l(\cos\theta) \end{aligned} \tag{13.70}$$

Thus, comparing (13.58) and (13.70), we find

$$a_l(k) = \frac{e^{i\delta_l}}{k}\sin\delta_l \tag{13.71}$$

Determining the scattering amplitude through a decomposition into partial waves is equivalent to determining the phase shift for each of these partial waves.

[3] In this section we have set the amplitude A of the incident plane wave equal to unity for mathematical simplicity. Note that the differential cross section (13.24) is independent of A.

In order to determine the total cross section

$$\sigma = \int d\Omega\,\frac{d\sigma}{d\Omega} = \int d\Omega\,|f(\theta)|^2 \tag{13.72}$$

we take advantage of (13.59) and the orthogonality of the spherical harmonics, namely,

$$\int d\Omega\, Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) = \delta_{l,l'}\delta_{m,m'} \tag{13.73}$$

to do the integral over the solid angle:

$$\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{13.74}$$

Comparing this result with the expression (13.70) for the scattering amplitude, we see that

$$\sigma = \frac{4\pi}{k}\,\mathrm{Im}\, f(0), \tag{13.75}$$

where we have taken advantage of the fact that the Legendre polynomials satisfy $P_l(1) = 1$. Equation (13.75) is known as the *optical theorem* and is a reflection of the fact that, as we discussed earlier, the very existence of scattering requires scattering in the forward direction in order to interfere with the incident wave and reduce the probability current in the forward direction.

Finally, it is common to express (13.74) as

$$\sigma = \sum_{l=0}^{\infty}\sigma_l \tag{13.76}$$

where

$$\sigma_l = \frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \tag{13.77}$$

the lth partial cross section, is the contribution to the total cross section by the lth partial wave. Note that the maximum value for the lth partial cross section occurs when the phase shift $\delta_l = \pi/2$.

13.5 EXAMPLES OF PHASE-SHIFT ANALYSIS

Hard-Sphere Scattering

We first analyze the scattering from the repulsive potential

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases} \tag{13.78}$$

hich characterizes a very hard (impenetrable) sphere. Our earlier discussion ggests that at sufficiently low energy the $l = 0$ partial wave dominates the pansion (13.70). Determining the phase shift is particularly easy for S-wave attering, since when $l = 0$ the radial equation simplifies considerably with the imination of the centrifugal barrier $l(l+1)\hbar^2/2\mu r^2$. Outside the sphere, the nction $u = Rr$ satisfies the free-particle equation

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} = Eu \qquad r > a \tag{13.79}$$

ather than write the solution to (13.79) in the form $u = B\cos kr + C\sin kr$ r in the form $u = Be^{ikr} + Ce^{-ikr}$), we are guided by asymptotic form for the dial wave function in (13.65) to write

$$u = C\sin(kr + \delta_0) \qquad r > a \tag{13.80}$$

here, as usual, $k = \sqrt{2\mu E/\hbar^2}$. Figure 13.10 shows a plot of u. The boundary ondition $u(a) = 0$ determines the S-wave phase shift:

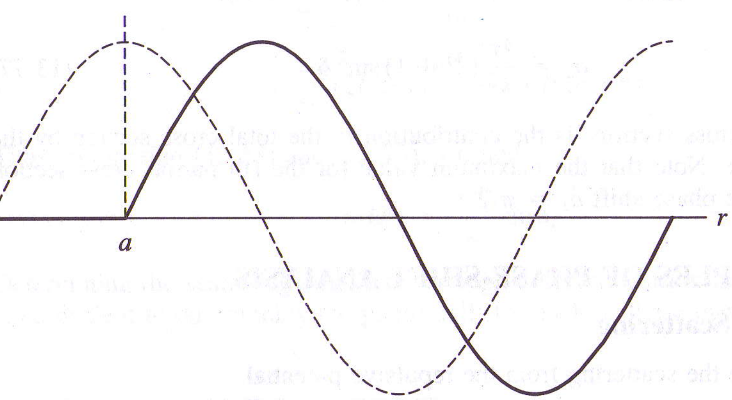
$$C\sin(ka + \delta_0) = 0 \qquad \text{or} \qquad \delta_0 = -ka \tag{13.81}$$

hus the S-wave total cross section is given by

$$\sigma_{l=0} = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2 ka \tag{13.82}$$

roblem 13.10 shows that the higher partial waves, such as the P-wave, can be eglected relative to the S-wave for hard-sphere scattering at low energy. Thus

$$\sigma \xrightarrow[ka\to 0]{} \sigma_{l=0} \xrightarrow[ka\to 0]{} \frac{4\pi}{k^2}(ka)^2 = 4\pi a^2 \tag{13.83}$$



IGURE 13.10
plot of the wave function $u = rR$ for S-wave scattering from a hard sphere showing the phase ift $\delta_0 = -ka$. The dashed curve shows u when $V = 0$.

Notice the cross section is indeed an area, but in this case the area is *four* times the classical cross section πa^2 that the sphere presents in the form of a disk that blocks the incident plane wave. Of course, low-energy scattering corresponds to a very long wavelength for the incident wave, and thus we should not expect to obtain the classical result. However, even at high energies and very short wavelengths we cannot completely avoid diffraction effects. We give a heuristic argument. At high energies, many partial waves, up to $l_{\max} = ka$, should contribute to the scattering. Therefore

$$\sigma \xrightarrow[ka\gg 1]{} \sum_{l=0}^{ka}\frac{4\pi}{k^2}(2l+1)\sin^2\delta_l \tag{13.84}$$

With so many l values contributing, we assume we can replace $\sin^2\delta_l$ by its average value, $\frac{1}{2}$, in the sum.[4] Then

$$\sigma \xrightarrow[ka\gg 1]{} \sum_{l=0}^{ka}\frac{2\pi}{k^2}(2l+1) \xrightarrow[ka\gg 1]{} 2\pi a^2 \tag{13.85}$$

Why do we obtain twice the classical result, even at high energy? For hard-sphere scattering, partial waves with impact parameters less than a must be deflected. However, in order to produce a "shadow" behind the sphere, there must be scattering in the forward direction (recall the optical theorem) to produce destructive interference with the incident plane wave. In fact, the interference is not completely destructive and the shadow has a bright spot (known in optics as Poisson's bright spot[5]) in the forward direction.

S-Wave Scattering for the Finite Potential Well

As another example of the determination of the phase shift at low energy, we examine scattering from an attractive potential, namely, the spherical well

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases} \tag{13.86}$$

In terms of the function $u = Rr$, the energy eigenvalue equation becomes

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} - V_0 u = Eu \qquad r < a \tag{13.87a}$$

$$-\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} = Eu \qquad r > a \tag{13.87b}$$

[4] For a detailed analysis, see J. J. Sakurai, *Modern Quantum Mechanics*, Benjamin-Cummings, Menlo Park, Calif., 1985, p. 408.
[5] Ironically, Poisson, who supported a corpuscular theory for light, refused to believe Fresnel's prediction that a bright spot would occur in the shadow of an illuminated disk, until it was experimentally verified.

quation (13.87a) can be written as

$$\frac{d^2u}{dr^2} = -k_0^2 u \qquad k_0 = \sqrt{\frac{2\mu}{\hbar^2}(E + V_0)} \qquad r < a \tag{13.88a}$$

hile equation (13.87b) is the usual

$$\frac{d^2u}{dr^2} = -k^2 u \qquad k = \sqrt{\frac{2\mu}{\hbar^2}E} \qquad r > a \tag{13.88b}$$

he solution to (13.88a) that satisfies the boundary condition $u(0) = 0$ is

$$u = A \sin k_0 r \qquad r < a \tag{13.89a}$$

s before, we write the solution outside the well in the form

$$u = C \sin(kr + \delta_0) \qquad r > a \tag{13.89b}$$

llowing explicitly for the appearance of a phase shift. The finite spherical well especially nice because we can determine analytically the wave function everyhere, both inside and outside the well.[6]

Making sure that u is continuous and has a continuous first derivative at $= a$, we obtain

$$A \sin k_0 a = C \sin(ka + \delta_0) \tag{13.90a}$$

$$A k_0 \cos k_0 a = C k \cos(ka + \delta_0) \tag{13.90b}$$

ividing these two equations, we obtain

$$\tan(ka + \delta_0) = \frac{k}{k_0} \tan k_0 a = \frac{ka}{k_0 a} \tan k_0 a \tag{13.91}$$

his equation for the S-wave phase shift simplifies considerably at sufficiently ow energy, that is, $ka \to 0$. Since $ka/k_0 a \ll 1$, as long as $\tan k_0 a$ is not too arge, the right-hand side of (13.91) is much less than one and we can replace he tangent of a small quantity with the quantity itself. Thus

$$ka + \delta_0 \cong \frac{ka}{k_0 a} \tan k_0 a, \tag{13.92a}$$

r

$$\delta_0 \cong ka\left(\frac{\tan k_0 a}{k_0 a} - 1\right) \tag{13.92b}$$

From (13.77),

$$\sigma_{l=0} = \frac{4\pi}{k^2} \sin^2 \delta_0 \cong \frac{4\pi}{k^2}\left[ka\left(\frac{\tan k_0 a}{k_0 a} - 1\right)\right]^2$$

$$= 4\pi a^2 \left(\frac{\tan k_0 a}{k_0 a} - 1\right)^2 \tag{13.93}$$

Since

$$k_0 a = \sqrt{k^2 a^2 + \frac{2\mu V_0 a^2}{\hbar^2}} \tag{13.94}$$

then for sufficiently small ka

$$k_0 a \cong \sqrt{\frac{2\mu V_0 a^2}{\hbar^2}} \tag{13.95}$$

and the total (S-wave) cross section is independent of energy.

Resonances

There is a significant exception to this independence of the cross section on energy. Suppose that the quantity $\sqrt{2\mu V_0 a^2/\hbar^2}$ is slightly less than $\pi/2$. Then as the energy increases, $k_0 a$, as given in (13.94), can reach the value of $\pi/2$. In this case, $\tan k_0 a$ is infinite, and therefore we can no longer assume that the right-hand side of (13.91) is small, even for small ka. In fact, at the value of the energy when $k_0 a = \pi/2$, $\tan(ka + \delta_0) = \infty$ and, consequently, $ka + \delta_0 = \pi/2$, which implies $\delta_0 \cong \pi/2$ since we are presuming $ka \ll 1$.[7] Thus

$$\sigma_{l=0} = \frac{4\pi}{k^2} \sin^2 \delta_0 \cong \frac{4\pi}{k^2} = 4\pi a^2 \left(\frac{1}{k^2 a^2}\right) \tag{13.96}$$

Here we see a pronounced dependence of the total cross section on energy. Also notice that the magnitude of the total cross section is much larger than that given in (13.93). Instead of being a small quantity, the phase shift $\delta_0 = \pi/2$.

What is causing this unusual behavior? If you return to our discussion of the bound states of the finite spherical well in Chapter 10, you will recognize the condition

$$\sqrt{\frac{2\mu V_0 a^2}{\hbar^2}} \cong \frac{\pi}{2} \tag{13.97}$$

[6] For other potential energies for which an analytic solution is not so easy to determine, we can still olve the energy eigenvalue equation by integrating the Schrödinger equation numerically outwards om the origin. Comparison of the numerical solution with the asymptotic form of the radial wave unction that appears in (13.65) permits a determination of the phase shift.

[7] Since the phase shift δ_0 starts at zero, we are assuming that the condition $\delta_0 = \pi/2$ is the first resonance condition that we reach as this phase shift grows.

as the condition that the well has a bound state at zero energy. Thus for a potential well satisfying (13.97), the energy of the scattering system is essentially the same as the energy of the bound state. In this case, the incident particle in a scattering experiment would like to form a bound state in the well. Since the system has a small but positive energy, the bound state isn't stable. However, pulling in the wave function in an attempt to form such a bound system dramatically changes the scattering behavior.

Although it is more difficult to determine the phase shifts and cross sections in such a nice analytic form as (13.92) for higher partial waves, it is easy to see physically why these higher partial waves may exhibit resonant behavior, with large "bumps" in the partial cross sections. These bumps arise when the phase shift goes through odd multiples of $\pi/2$. Figure 13.11 shows a plot of the effective potential energy

$$V_{\text{eff}}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2} \tag{13.98}$$

for the spherical well. A particle with energy E greater than zero but less than the height of the barrier can tunnel through the barrier and form a metastable bound state in the well. This state is metastable (and not stable) because a particle "trapped" inside the well can also tunnel out. Thus if the energy of the beam in a scattering experiment is tuned to one of the energies of these metastable states, there is an enhanced tendency for the particle to get stuck in the well. The system then loses track of the mechanism by which the bound state was formed, that is, in particular it may lose track of the direction of the incident beam. When this metastable state decays, it emits the particle with the characteristic angular distribution for that particular decay mode.

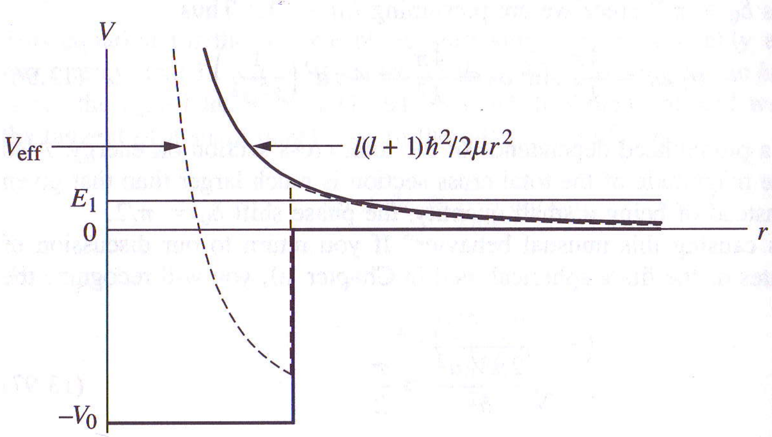


FIGURE 13.11
The centrifugal barrier combines with the potential well to produce an effective potential that can produce a metastable state, as indicated. If the energy of the incident beam coincides with the energy of one of these metastable states, a resonance in the scattering cross section can occur.

A convenient way to parametrize the behavior of the phase shift in the vicinity of a resonance leads to the famous Breit-Wigner formula. We assume the phase shift δ_l of the lth partial wave goes through $\pi/2$ at an energy E_0:

$$\delta_l(E_0) = \frac{\pi}{2} \tag{13.99}$$

We next make a Taylor series expansion of $\cot\delta_l$ in the vicinity of the resonant energy:

$$\begin{aligned} \cot\delta_l(E) &= \cot\delta_l(E_0) + \left(\frac{d\cot\delta_l}{dE}\right)_{E=E_0}(E-E_0) + \cdots \\ &= -\left(\frac{1}{\sin^2\delta_l}\frac{d\delta_l}{dE}\right)_{E=E_0}(E-E_0) + \cdots \end{aligned} \tag{13.100}$$

Defining

$$\left(\frac{d\delta_l(E)}{dE}\right)_{E=E_0} = \frac{2}{\Gamma} \tag{13.101}$$

we obtain

$$\cot\delta_l(E) = -\frac{2}{\Gamma}(E-E_0) + \cdots \tag{13.102}$$

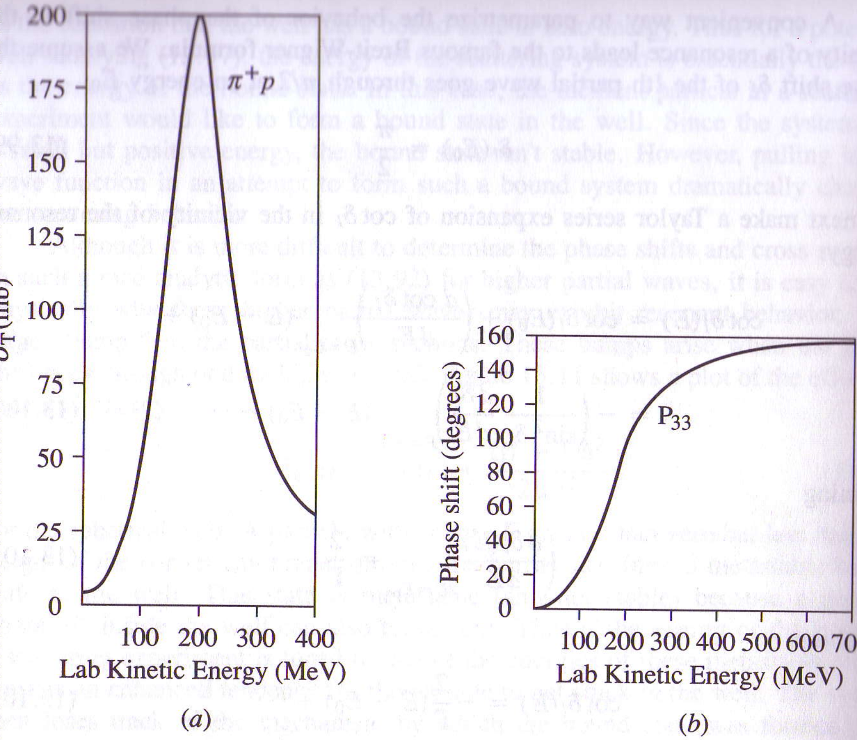
Finally, we can express the function $a_l(k)$ (see 13.71) as

$$\begin{aligned} a_l(k) &= \frac{e^{i\delta_l}}{k}\sin\delta_l \\ &= \frac{1}{k}\frac{1}{\cot\delta_l - i} \\ &\cong \frac{1}{k}\frac{1}{-(2/\Gamma)(E-E_0) - i} = -\frac{1}{k}\frac{\Gamma/2}{(E-E_0)+i\Gamma/2} \end{aligned} \tag{13.103}$$

Repeating the steps leading to (13.74), we find that the total cross section for the lth partial wave in the vicinity of a resonance is thus given by

$$\sigma_l \cong \frac{4\pi}{k^2}(2l+1)\frac{\Gamma^2/4}{(E-E_0)^2+\Gamma^2/4} \tag{13.104}$$

As an example, Fig. 13.12*a* shows a strong resonance in $\pi^+ - p$ scattering with a peak at roughly 190 MeV of incident pion kinetic energy. This resonance, known as the $\Delta(1232)$ because the center-of-mass energy of the resonance at the peak is 1232 MeV, has a full width at half maximum Γ of 110–120 MeV. Figure 13.12*b* shows the P-wave phase shift, which reaches $\pi/2$ at the resonance peak. The resonance is thus formed in the $l = 1$ channel. In fact, the intrinsic spin of the $\Delta(1232)$ is $j = \frac{3}{2}$. From (13.104), we expect $\sigma_{l=1} = 12\pi/k^2$



IGURE 13.12
$\text{z})$ The total cross section for $\pi^+ - p$ scattering with pion kinetic energies up to 400 MeV *(From . R. Frazer,* Elementary Particles, *Prentice-Hall, Englewood Cliffs, N. J., 1966.) (b)* The P-wave hase shift for $\pi^+ - p$ scattering. *(From L. D. Roper, R. M. Wright, and B. T. Feld,* Phys. Rev. **38**, *B190, 1965.)*

hen $E = E_0$. However, when we add the orbital angular momentum $l = 1$ of e pion-proton system to an intrinsic spin $s = \frac{1}{2}$ of the proton, we can form a tal angular momentum $j = \frac{1}{2}$ (two states) as well as $j = \frac{3}{2}$ (four states). Thus of e six total angular momentum states that are generated (presumably incoherently) the collision, only the four $j = \frac{3}{2}$ states can produce the resonance. If we ssume the scattering cross section for the $j = \frac{1}{2}$ states is negligible in comparison ith the $j = \frac{3}{2}$ resonance scattering in the vicinity of the peak, the $\pi^+ - p$ total oss section at the resonance should be $\frac{4}{6}$ of the P-wave peak cross section, that $8\pi/k^2$, which is about 190 mb, in good agreement with the observed value.

Finally, we return to our discussion of S-wave scattering for the finite poten- al well and ask what happens if we increase the energy of the beam so that the hase shift $\delta_0 \to \pi$, as illustrated in Fig. 13.13. Then the wave function outside e well,

$$u = C\sin(kr + \pi) = -C\sin kr \tag{13.105}$$

the same as the wave function outside the well with zero phase shift [see

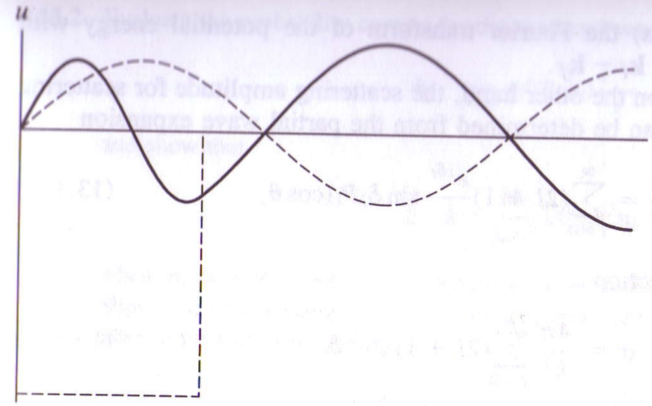


FIGURE 13.13
The wave function $u(r)$ for S-wave scattering for a potential well at an energy such that the phase shift $\delta_0 = \pi$. The dashed curve shows u when $V = 0$.

(13.89*b*)] up to an *overall* phase. Moreover, since $\sin\delta_0 = 0$, the S-wave partial cross section vanishes ($\sigma_{l=0} = 0$). This effect, known as the Ramsauer-Townsend effect,[8] is clearly seen in the very low scattering cross section from noble gases at about 0.7 eV. These rare-gas atoms have a potential well with a sharply defined range, and it is possible at low energies to have $\delta_0 = \pi$, with all other phase shifts negligible, leading to essentially perfect transmission of the incident wave.

13.6 SUMMARY

The differential scattering cross section is given by

$$\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2 \tag{13.106}$$

where the scattering amplitude $f(\theta,\phi)$ determines the angular dependence in the asymptotic form for the solution to the Schrödinger equation:

$$\psi \xrightarrow[r\to\infty]{} Ae^{ikz} + Af(\theta,\phi)\frac{e^{ikr}}{r} \tag{13.107}$$

At "high" energies, the scattering amplitude can be determined through the Born approximation

$$f(\theta,\phi) = -\frac{\mu}{2\pi\hbar^2}\int d^3\mathbf{r}'\, V(r')e^{i\mathbf{q}\cdot\mathbf{r}'} \tag{13.108}$$

[8] Unfortunately, a different Townsend.

ich is (up to constants) the Fourier transform of the potential energy with
pect to the vector $\mathbf{q} = \mathbf{k}_i - \mathbf{k}_f$.

At "low" energies, on the other hand, the scattering amplitude for scattering
m a central potential can be determined from the partial wave expansion

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{e^{i\delta_l}}{k} \sin\delta_l P_l(\cos\theta) \tag{13.109}$$

ding to a total cross section

$$\sigma = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2\delta_l \tag{13.110}$$

e phase shifts δ_l, which are generated when the particular partial waves interact
th the potential, enter in the asymptotic expression for the wave function

$$\psi(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} \sum_{l=0}^{\infty} C_l \frac{\sin[kr - l\pi/2 + \delta_l(k)]}{kr} P_l(\cos\theta) \tag{13.111}$$

useful way to determine the phase shifts is to solve the Schrödinger equation,
her numerically or analytically, for the radial wave function $u = rR$, which
eys the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2\mu} \frac{d^2u}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} u + V(r)u = Eu \tag{13.112}$$

d compare the asymptotic form of the solution with (13.111).

ROBLEMS

3.1. Use the three-dimensional time-dependent Schrödinger equation

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

to establish that the probability density $\psi^*(\mathbf{r}, t)\psi(\mathbf{r}, t)$ obeys the local conservation law

$$\frac{\partial}{\partial t}(\psi^*\psi) + \nabla\cdot\mathbf{j} = 0$$

where

$$\mathbf{j} = \frac{\hbar}{2\mu i}(\psi^*\nabla\psi - \psi\nabla\psi^*)$$

What would happen to your derivation if the potential energy V were imaginary? Is probability conserved? Explain. In nonrelativistic quantum mechanics, such an imaginary potential energy can be used, for example, to account for particle absorption in interactions with the nucleus.

13.2. Evaluate the probability current for the scattered wave

$$\psi_{sc} \underset{r\to\infty}{\longrightarrow} A f(\theta,\phi)\frac{e^{ikr}}{r}$$

and show that

$$\mathbf{j}_{sc} \underset{r\to\infty}{\longrightarrow} \frac{\hbar k}{\mu r^2}|A|^2|f|^2\mathbf{u}_r$$

where $\mathbf{u}_r$ is a unit vector in the direction of the radius.

13.3. Show that at low energies ($k \to 0$) the requirement (13.48) for the validity of the Born approximation becomes

$$\left|\frac{2\mu}{\hbar^2 k}\int_0^\infty dr'\,V(r')kr'\right| \sim \frac{\mu V_0 a^2}{\hbar^2} \ll 1$$

where V_0 is the order of magnitude of the potential energy, a is the range of the potential, and we have neglected constants of order unity. By comparing this result with (13.49), argue that if the Born approximation is valid at low energies, it works at high energies as well.

13.4. Using the Born approximation to determine the one-dimensional reflection coefficient R for a potential energy $V(x)$ that vanishes everywhere except in the vicinity of the origin:

(*a*) Show that we can write the solution to the one-dimensional Schrödinger equation in the form

$$\psi(x) = Ae^{ikx} + \int dx'\,G(x,x')\frac{2m}{\hbar^2}V(x')\psi(x')$$

where

$$\frac{\partial^2}{\partial x^2}G(x,x') + k^2 G(x,x') = \delta(x-x')$$

(*b*) Since G satisfies a second-order differential equation, G must be a continuous function and, in particular, it must be continuous at $x = x'$. By integrating the differential equation for G from just below to just above $x = x'$, show that the first derivative of G is discontinuous at $x = x'$ and that it satisfies

$$\left(\frac{\partial G}{\partial x}\right)_{x=x'_+} - \left(\frac{\partial G}{\partial x}\right)_{x=x'_-} = 1$$

Then show that one solution for G is given by

$$G = \begin{cases} \dfrac{1}{2ik}e^{ik(x-x')} & x > x' \\ \dfrac{1}{2ik}e^{-ik(x-x')} & x < x' \end{cases}$$

(*c*) Substitute this expression for G into the equation for ψ in (*a*). Show that in the Born approximation

$$\psi \xrightarrow[x\to-\infty]{} Ae^{ikx} + Ae^{-ikx}\int_{-\infty}^{\infty} dx'\,\frac{e^{2ikx'}}{2ik}\frac{2m}{\hbar^2}V(x')$$

and that consequently

$$R = \left|\frac{m}{ik\hbar^2}\int_{-\infty}^{\infty} dx'\, e^{2ikx'}V(x')\right|^2$$

(*d*) For the potential barrier

$$V(x) = \begin{cases} V_0 & 0 < x < a \\ 0 & \text{elsewhere} \end{cases}$$

the exact reflection coefficient is given by $R = 1 - T$ with

$$T = \frac{1}{1 + [V_0^2/4E(E - V_0)]\sin^2\sqrt{(2m/\hbar^2)(E - V_0)}a}$$

Show that the exact result for R in the limit $V_0/E \ll 1$ agrees with the result of the Born approximation.

13.5. In the initial Rutherford scattering experiment Geiger and Marsden used α particles with an energy of 5 MeV. Choose a reasonable value for m_0 and determine the range of angles for which (13.56) should be valid. *Suggestion:* Write $m_0 = 1/a$, where a is a characteristic screening length. How large should a be? *Note:* Interaction with the *electrons* in the atom produces a deflection on the order of 10^{-4} radians.

13.6. Use the Born approximation to determine the differential cross section for the potential energy

$$V = \frac{C}{r^2}$$

where C is a constant, corresponding to a $1/r^3$ force. *Note:* The result depends on $\hbar$, so it is not the same as the classical result.

13.7. Use the Born approximation to determine the differential cross section for the potential energy

$$V(r) = V_0 e^{-r/a}$$

13.8. Our goal is to establish (13.60), which, with the aid of (13.59), can be put in the form

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} i^l\sqrt{4\pi(2l+1)}\, j_l(kr)Y_{l,0}(\theta)$$

(*a*) Explain why the expansion of the plane wave must be of the form

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty} c_l j_l(kr)Y_{l,0}(\theta)$$

(*b*) Use the fact that

$$|l,0\rangle = \frac{1}{\sqrt{(2l)!}}\left(\frac{\hat{L}_-}{\hbar}\right)^l |l,l\rangle$$

(see Problem 9.18) to show

$$c_l j_l(kr) = \frac{1}{\sqrt{(2l)!}}\int d\Omega\, Y_{l,l}^*\left[\left(\frac{1}{i}e^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right)\right)^l e^{ikr\cos\theta}\right]$$

$$= \frac{1}{\sqrt{(2l)!}}\int d\Omega\, Y_{l,l}^*\left[(-1)^l e^{il\phi}\sin^l\theta\frac{d^l}{d(\cos\theta)^l}e^{ikr\cos\theta}\right]$$

where the last step follows from the explicit form (9.142) for the raising operator in position space.

(*c*) Use the explicit form (9.146) for $Y_{l,l}(\theta,\phi)$ to express this result in the form

$$c_l j_l(kr) = (ikr)^l\frac{2^l l!}{\sqrt{(2l)!}}\sqrt{\frac{4\pi}{(2l+1)!}}\int d\Omega\,|Y_{l,l}(\theta,\phi)|^2 e^{ikr\cos\theta}$$

(*d*) Finally, isolate c_l by evaluating this expression as $r \to 0$. *Hint:* For small r, $j_l(kr)$ behaves as $(kr)^l/(2l+1)!!$, where $(2l+1)!! = (2l+1)(2l-1)\ldots5\cdot3\cdot1$.

13.9. A particle is scattered by a spherically symmetric potential at sufficiently low energy that the phase shifts $\delta_l = 0$ for $l > 1$ (that is, only δ_0 and δ_1 are nonzero). Show that the differential cross section has the form

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta + C\cos^2\theta$$

and determine A, B, and C in terms of the phase shifts. Determine the total cross section σ in terms of A, B, and C.

13.10. Evaluate the P-wave phase shift δ_1 for scattering from a hard sphere, for which the potential energy is given by

$$V(r) = \begin{cases} \infty & r < a \\ 0 & r > a \end{cases}$$

Express your result in terms of $j_1(ka)$ and $\eta_1(ka)$. Use the leading behavior of $j_1(\rho)$ and $\eta_1(\rho)$ for small ρ to show that $\delta_1 \to -(ka)^3/3$ as $ka \to 0$, and thus δ_1 can indeed be neglected in comparison to δ_0 [see (13.81)] at sufficiently low energy.

13.11. Compare the Born approximation result for the total cross section for scattering from the potential well

$$V(r) = \begin{cases} -V_0 & r < a \\ 0 & r > a \end{cases}$$

with that obtained by using S-wave phase shift analysis. Using the condition for the validity of the Born approximation at low energy (see Problem 13.3), show that the two approaches are in agreement when the Born approximation is valid.

12. Consider the spherically symmetric potential energy $2\mu V(r)/\hbar^2 = \gamma\delta(r-a)$, where γ is a constant and $\delta(r-a)$ is a Dirac delta function that vanishes everywhere except on the spherical surface specified by $r = a$.

(*a*) Show that the S-wave phase shift δ_0 for scattering from this potential satisfies the equation

$$\tan(ka + \delta_0) = \frac{\tan ka}{1 + (\gamma/k)\tan ka}$$

(*b*) Evaluate the phase shift in the low-energy limit and show that the total cross section for S-wave scattering is

$$\sigma \cong 4\pi a^2 \left(\frac{\gamma a}{1+\gamma a}\right)^2$$

13 (*a*) Determine the differential cross section $d\sigma/d\Omega$ in the Born approximation for scattering from the potential energy $2\mu V(r)/\hbar^2 = \gamma\delta(r-a)$ (see Problem 13.12). Show the explicit dependence of $d\sigma/d\Omega$ on θ.

(*b*) Evaluate $d\sigma/d\Omega$ in the low-energy limit. Show that the differential cross section is isotropic. What is the total cross section?

(*c*) Use the condition for the validity of the Born approximation at low energy (see Problem 13.3) to establish that your result in (*b*) for the total cross section agrees with that given in Problem 13.12 in the appropriate limit.

CHAPTER 14

PHOTONS AND ATOMS

In this chapter we turn our attention to a quantum treatment of the electromagnetic field. After analyzing the Aharonov-Bohm effect, which demonstrates the unusual role played by the vector potential in quantum mechanics, we use the vector potential to show that the Hamiltonian for the electromagnetic field can be expressed as a collection of harmonic oscillators. The raising and lowering operators for these oscillators turn out to be creation and annihilation operators for photons, the quanta of the electromagnetic field. This quantum theory of the electromagnetic field is then used to determine the lifetimes of excited states of the hydrogen atom using time-dependent perturbation theory.

14.1 THE AHARONOV-BOHM EFFECT

Within classical physics, the vector potential **A** is simply an auxiliary field that is introduced to help determine the physical electromagnetic fields **E** and **B**. In particular, Gauss's law for magnetism,

$$\nabla \cdot \mathbf{B} = 0 \tag{14.1}$$

implies that we can write

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{14.2}$$