1 Systems of Linear Equations

Many problems that occur naturally involve finding solutions that satisfy systems of linear equations of the form

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n & = & b_n \end{array}$$

involving different variables, $x_1, x_2, x_3, \cdots, x_n$ and constants b_i for $1 \leq i \leq n$ and a_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq n$. Such problems involve finding a values for the variables x_i that satisfy simultaneously satisfy each of the equations. In the case of two equations in two variables this presents few difficulties, but as the number of equations and the number of variables increase, the complexity increases enormously. It is not uncommon for problems to arise in which one seeks solutions to a system of equations involving hundreds of equations and variables. The complexity, both theoretical and computational, can be daunting. Here our goals are modest. We shall examine systems of equations involving not more than 3 equations in three variables. More advanced techniques are covered in a course in Linear Algebra. Before engaging directly in the topic, we will take a brief detour and talk about three dimensional Cartesian geometry, in particular the equations of planes . This will help in the understanding of the subtleties of the task.

1.1 Three dimensional cartesian geometry

So far we have been working in two dimensional Cartesian geometry, named after René Descartes (1596-1650) the famous French philosopher who got the whole thing started. Here we have chosen two copies of the *number line* that intersect at right angles and all points in the plane are located according to distances along these two lines. The point of intersection corresponds to the number zero on each number line. Each point in the plane is then represented by a pair of numbers (x, y) where the first number, x, represents distance along the so-called *x-axis* and the second, y, distance along the *y-axis*. Furthermore these lines have been chosen with a certain *orientation* in mind. By this I mean, if one were to place ones right-hand on the plane with your wrist at the origin and your fingers pointing towards the positive x-axis , then as you curl your fingers, they begin to point toward the positive y-axis. The reverse orientation would be if you curled fingers pointed to the negative y-axis. This model of the plane is referred to as the Cartesian plane. We know from work done earlier that any line in the Cartesian plane can be represented by an equation of the form $ax + by = c$ for some constants a, b, c.

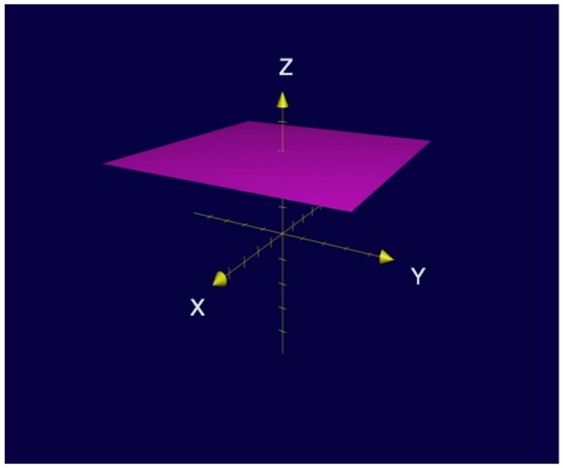

Figure 1: Arrows point in the direction of the positive x, y, and z axes. The magenta plane is the graph of the equation $z = 3$.

For three dimensional Cartesian geometry we choose not two but three mutually perpendicular copies of the number line which intersect one another at the point zero on each of the lines. The lines now are referred to as the x, y and z axes, and points in three-dimensional space are represented as a triple of numbers (x, y, z), where the first represents distance along the *x- axis* the second distance along the *y-axis,* and the third along the *z-axis.* The so called *xy-plane* consists of those points of the form $(x, y, 0)$ where the z coordinate is zero. The lines have again been chosen with a certain orientation so that as before if one were to place ones right hand on the xy-plane with wrist at the origin and fingers pointing toward the positive x axis, then keeping the thumb extended, as one curls the fingers toward the positive y axis , the thumb should be pointing in the direction of the positive z axis. See figure 1.

Just as any linear equation $ax + by = c$ represents a line in 2-dimensional space, a linear equation of the form $ax + by + cz = d$ represents a plane in 3 dimensional space. In figure 1 the magenta plane is the graph of the equation $z = 3$. It forms the collection of all points of the form $(x, y, 3)$, for the only restriction that the equation describes is that the z coordinate must equal 3. We also observe that this plane is parallel to the xy-plane consisting of all points of the form $(x, y, 0)$.

In a similar way, the so called *xz-plane* consists of all the points of the form $(x, 0, z)$ and the *yz-plane* consists of all points of the form $(0, y, z)$. Collectively the three planes, the xy, the xz, and the yz are referred to as the *coordinate planes.* As we see in figures 2 and 3, the equations $x = 3$ and $y = 3$ have graphs that are parallel to respectively the yz-plane and the xz-plane. In figure 4 we show the graph of the equation $x + y + z = 1$, which in

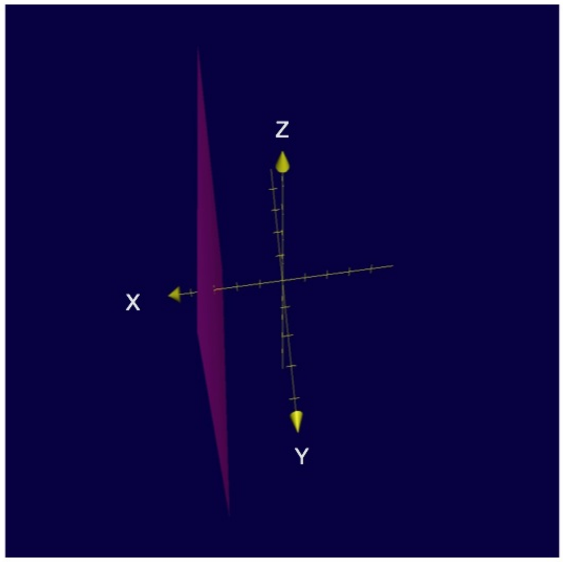

Figure 2: Magenta plane as the graph of $x = 3$

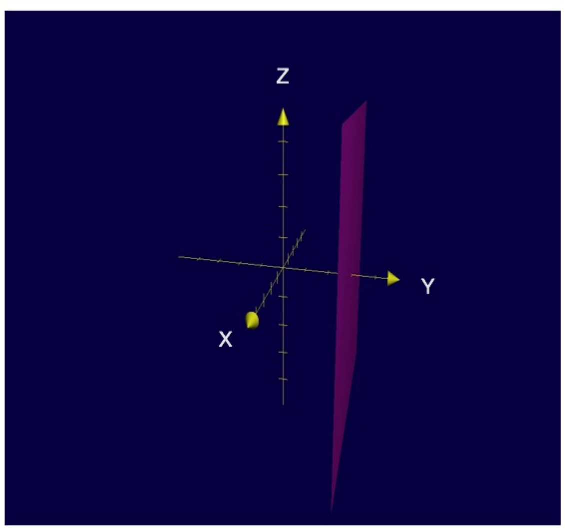

Figure 3: Magenta plane as the graph of $y = 3$

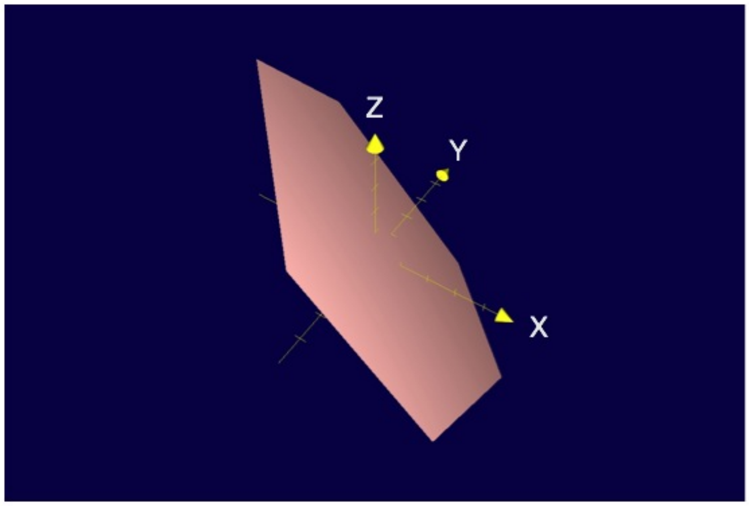

Figure 4: Graph of the equation $x + y + z = 3$.

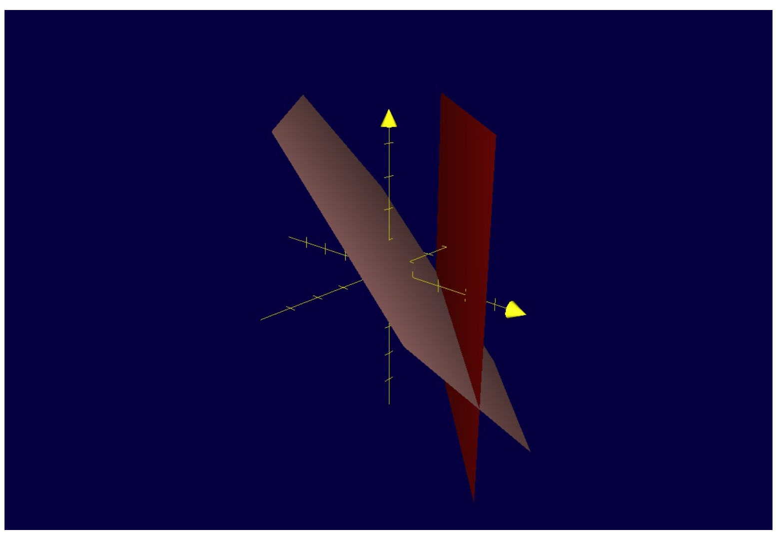

Figure 5: Plane with equation $x + y + z = 1$ intersects with that with equation $x + y = 3$

particular passes through the points $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

In two dimensions, two lines intersect in a point, but in three dimensions, 2 planes, if they intersect, the intersection is along a complete line. The points on this line then satisfy the equations of each of the two planes. In figure 5 , the plane with equation $x + y + z = 1$ is seen to intersect with the equation $x + y = 3$. Now what about three planes? Given two of the planes, if they intersect, they intersect in a line. Given 3 planes, several situations may occur

- all three planes may be parallel , in which case there are no points common to all three planes; see figure 6.
- the three planes may intersect in a common line in which case any point on the line is a common to all three planes; see figure 7.
- two of the planes may intersect in a common line l but the third plane although it intersects each of the first two does not intersect l. In this case also there are no points common to all three planes; see figure 8.
- two of the planes may intersect in a line l but the third is parallel to one of the first two and therefore does not intersect l. In this case again there are no points common to all of the three planes; see figure. 9.
- the three planes may intersect in a unique common point; see figure 10.

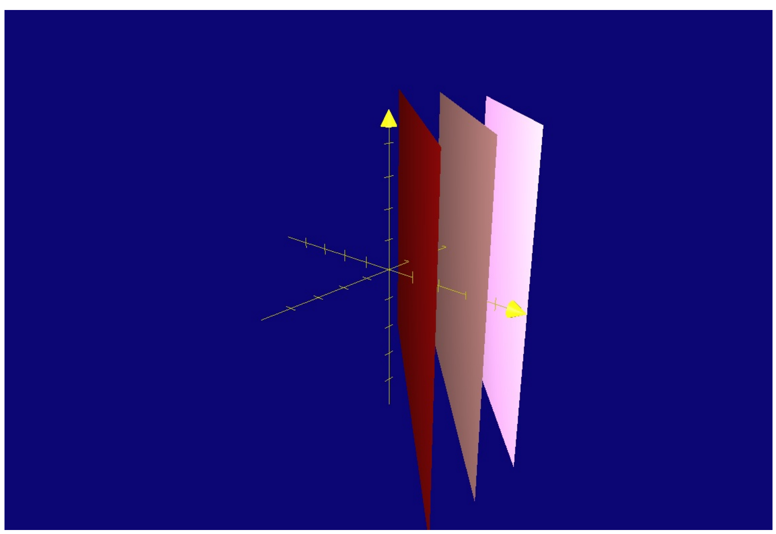

Figure 6: Three parallel planes

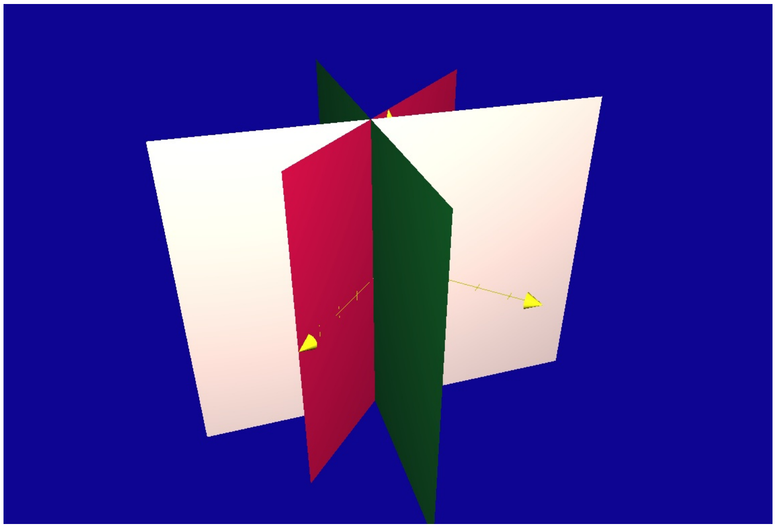

Figure 7: Three planes intersecting in a common line

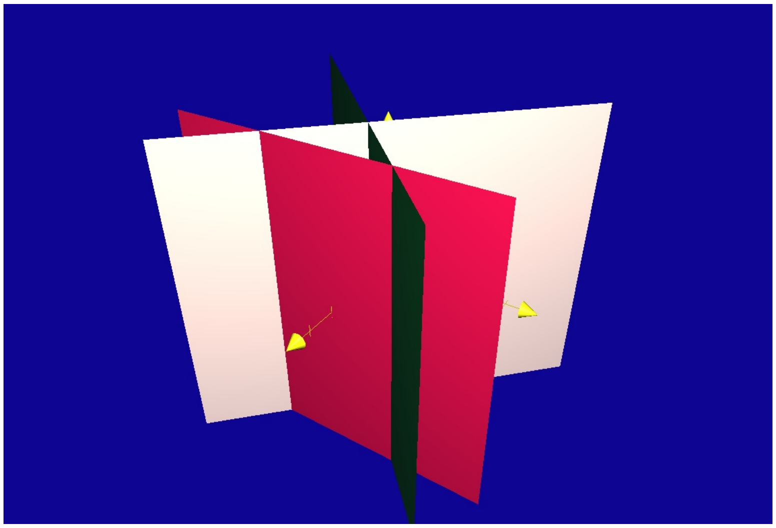

Figure 8: Planes $x - y = 1$, $x + y = 1$, and $x = 3$ with no points common to all three.

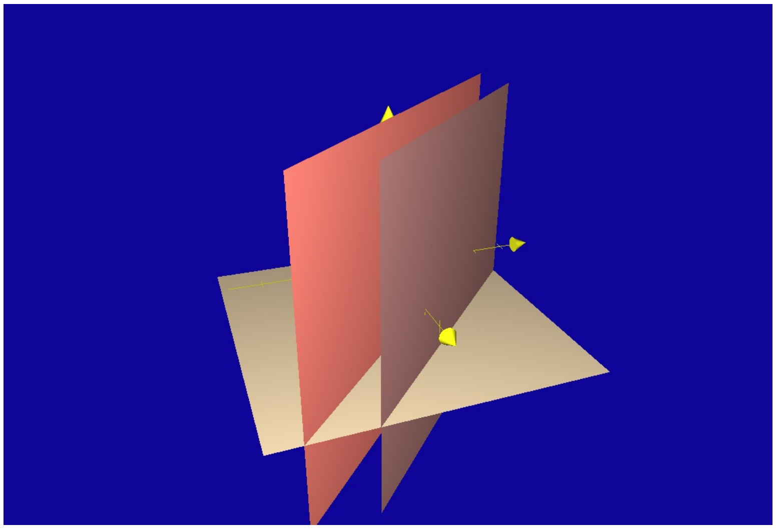

Figure 9: Two parallel planes with equations $x + y = 1$ and $x = y = 3$ intersected with a third $z = -2$ creating two lines of intersection with no points common to all three planes

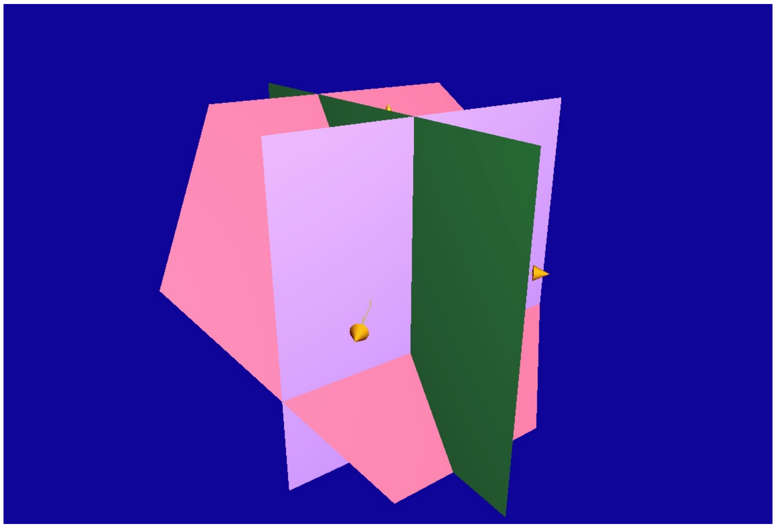

Figure 10: Three planes with equations $x + y + z = 1, x + y = 3$ and $x - y = 1$ intersecting in a unique point

1.2 Solutions to sets of equations

The situation of two equations in two variables presents only a few complications. We begin with a review of the necessary techniques. The situation of three equations in three variables is somewhat more complicate and the the techniques one uses to find solutions can be generalized to yet more complicated systems.

1.3 Two equations in two two variables

If we are given two linear equations, $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, then each also represents a line in 2 dimensional Cartesian space with each point on the line representing a solution to the respective equation. If the two lines intersect then there is a unique point on both lines and hence a simultaneous solution to both equations. The only situation in which the lines do not intersect is when they are parallel, and in this case there are no values of x and y that satisfy both equations. For instance if we are given two the equations $x + y = 1$ and $x + y = 3$, the corresponding lines are parallel and there is no simultaneous solution.

The other case in which two linear equations as above may have no unique solution is when one equation is simply a rewrite of the other - so that geometrically we have only one line. For instance if the first equation is $x+y = 1$ and the second is $2x+2y = 2$, then the second is obtained by multiplying the first by 2.

However, for the remaining case in which one equation is not a multiple of the or in which the associated lines are mutually parallel, there the unique point of intersection may be found by solving for one of the variables in one of the equations and then substituting the result into the second. This technique works even if the two equations are not necessarily linear.

Example 1 *Professor Fan started a small business selling his famous Yin-yang soft drink as an alternative to Pepsi. His initial equipment investment was $15,000. Each bottle cost $ 0.75 to produce and it sold for $ 1.50. Ignoring other costs of the business for which he had alternative financing , how many bottles of Ying-yang drink must be sold in order to pay off the investment in equipment?*

Solution: If x represents the number of bottles, we have two equations, an equation describing the cost of producing x bottles , which is $y = 0.75x + 15,000$ and a revenue equation, $y = 1.50x$. This is a simple example, because we have already solved for the y variable in both equations. So taking the expression for y in the second equation and substituting it into the first, we get, $1.50x = 0.75x + 15,000$. Solving for x we have $x = \frac{15,00}{0.75} = 15,000\frac{4}{3} = 20,000$. ■

Example 2 *A local shop packaged a mixture of cashews and almonds which was sold in 1 pound bags for \$6 each. The store was able to buy the cashews for \$ 6 a pound and the almonds for \$3 a pound. How many pounds of cashews and almonds made up the 1 pound mixture if the store realized a profit of \$1 for each bag sold?*

Solution: Let x denote cashews and y the amount of almonds in the mixture. Then $x + y = 1$. The cost to the store of a 1 pound bag is \$5. Thus $6x + 3y = 5$. We thus have a system of two equations in two variables. Solving for y in the first gives $y = 1 - x$, and substituting this into the second equation gives $6x + 3(1 - x) = 5$ or $3x = 2$. Thus $x = \frac{2}{3}$, and $y = \frac{1}{3}$. ■

Example 3 *Solve the system of equations*

$$\left\{ \begin{array}{rr} 2x - 3y = & -8 \\ 14x - 21y = & 3 \end{array} \right\}$$

Solution: Solving for x in the first equation gives $x = \frac{3}{2}y - 4$. Substituting into the second equation gives $14(\frac{3}{2}y - 4) - 21y = 3$, or $-56 + 21y - 21y = 3$ or equivalently $-56 = 3$ which is clearly absurd. So what went wrong? Do the two equations represent parallel lines? The easiest way to answer this is to put both in the standard form $y = mx + b$ and then to see if the value of m, the slope is the same in both cases. Doing this we can rewrite the equations as

$$\left\{ \begin{array}{rl} y = & \frac{2}{3}x + 3 \\ y = & \frac{2}{3}x - \frac{1}{7} \end{array} \right\}$$

So the answer to the problem is that there are no solutions and that the two equations represent parallel lines. ■

1.4 Three equations in three variables

Given a system of three equations in three variables

$$\left\{ \begin{array}{rl} a_1x + b_1y + c_1z = & d_1 \\ a_2x + b_2y + c_2z = & d_2 \\ a_3x + b_3y + c_3z = & d_3 \end{array} \right\}$$

as we have seen each of these equations represents a plane in 3 dimensional Cartesian space and that a unique solutions exists only when the three planes intersect in a unique point. All this is wonderful, but how does it help us solve a particular system. The technique of solving for one variable in terms of another which is used for two equations can be used but often leads to confusion. There is a tried and true technique that never fails and which leads to an easy expression of solutions in the case that one all three planes intersect in a common line. The technique is referred to as *Gaussian elimination,* named after the famous 19^{th} century mathematician Carl Friedrich Gauss (1777-1855). The technique relies on four simple observations.

1. the order of the equations can be changed
2. any equation of the system can be multiplied by a constant on both the left and the right and the solutions to the equation (the points on the determined plane) are not changed and therefore the system itself is not changed
3. one equation may be added to another with out changing the solutions to the system
4. a multiple of one row may be added to another without changing the solutions to the system

The fourth item follows from the second and the third, but the third perhaps needs some explanation. Suppose x, y, and z are numbers that satisfy all three equations above and lets look at the system in which the first equation has been added to the second. This system is

$$\left\{\begin{array}{rr} a_1x+b_1y+c_1z= & d_1 \\ (a_2x+b_2y+c_2z)+(a_1x+b_1y+c_1z)= & d_2+d_1 \\ a_3x+b_3y+c_3z= & d_3 \end{array}\right\}$$

Since x, y, z satisfy the first equation in the second system , we have added the same number to both sides of the second equation to form the second system. This number can then be cancelled from both sides, and we are back to the first system. Thus the second system reduces to the first.

With these observations in mind the algorithm for solving a system of three equations with three variables follows a sequence of procedures. We will illustrate the method by solving the following system

$$\left\{\begin{array}{rr} 2y+4z= & 10 \\ 2x-4y+6z= & 8 \\ -x+3y= & 12 \end{array}\right\}$$

We see from figure 11 , that the three planes intersect in a single point so we are guaranteed a unique solution of the system of equations.

Step 1: Rewrite the equations, if necessary, so that if one of the variables does not occur in one of the equations, it is then inserted with a coefficient of zero. That is:

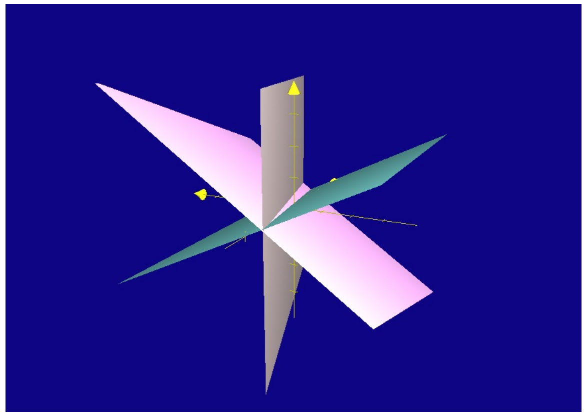

Figure 11: Looking down the negative x axis, the intersection of three planes, $2x-4y+6z=8$ in pink, $2y+4z=10$ in blue-green, and $-x+3y=12$ in brown.

$$x+y=3$$

becomes

$$x+y+0z=3$$

Step2: Make sure that all equations show from left to right first the x, then the y and then the z varaiables. Also

Step 3: Delete the variable symbols, the plus and equal signs leaving only an array of numbers called a matrix. For instance, in the case of the system of equations mentioned above, applying steps 1, 2, and 3 ,we get the following array

$$\begin{pmatrix} 0 & 2 & 4 & | & 10 \\ 2 & -4 & 6 & | & 8 \\ -1 & 3 & 0 & | & 12 \end{pmatrix}$$

The vertical lines in the fourth column are a notational convention and strictly speaking not necessary. They are there simply to remind us that there once had been equal signs in their place. The observations 1, 2 and 4 above can be interpreted as operations on the rows of the matrix. These are called *elementary row operations*. In detail they are:

- interchange row i with row j - written for convenience as R_i, R_j o
- multiply row i by a constant k - written as kR_i
- add k times row i to row j - written as written $kR_i + R_j$

Step 4: Interchange rows so that the first row has a non-zero term in the first position.

$$\begin{pmatrix} 0 & 2 & 4 & | & 10 \\ 2 & -4 & 6 & | & 8 \\ -1 & 3 & 0 & | & 12 \end{pmatrix} \xrightarrow[R_1, R_3]{} \begin{pmatrix} -1 & 3 & 0 & | & 12 \\ 2 & -4 & 6 & | & 8 \\ 0 & 2 & 4 & | & 10 \end{pmatrix}$$

Step 5: If the first entry in the first row is not 1, multiply the first row by the reciprocal of the entry in the first row using an elementary operation of type 2

$$\begin{pmatrix} -1 & 3 & 0 & | & 12 \\ 2 & -4 & 6 & | & 8 \\ 0 & 2 & 4 & | & 10 \end{pmatrix} \xrightarrow[-1R_1]{} \begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 2 & -4 & 6 & | & 8 \\ 0 & 2 & 4 & | & 10 \end{pmatrix}$$

Step 6: If necessary, add a suitable multiple of the first row to the second and third rows so that the first position in the second and third rows becomes zero.

$$\begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 2 & -4 & 6 & | & 8 \\ 0 & 2 & 4 & | & 10 \end{pmatrix} \xrightarrow[-2R_1 + R_2]{} \begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 0 & 2 & 6 & | & 32 \\ 0 & 2 & 4 & | & 10 \end{pmatrix}$$

Step 7: Interchange rows 3 and 4 if necessary, to make sure that second entry in the second row does not contain all zeros. Repeat steps 4, 5, and 6 with only the last two rows while neglecting the first column. Then repeat step 5 with the last row while neglecting the first two columns.

$$\begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 0 & 2 & 6 & | & 32 \\ 0 & 2 & 4 & | & 10 \end{pmatrix} \xrightarrow[\frac{1}{2}R_2]{} \begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 0 & 1 & 3 & | & 16 \\ 0 & 2 & 4 & | & 10 \end{pmatrix} \xrightarrow[-2R_2 + R_3]{} \begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 0 & 1 & 3 & | & 16 \\ 0 & 0 & -2 & | & -22 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 0 & 1 & 3 & | & 16 \\ 0 & 0 & -2 & | & -22 \end{pmatrix} \xrightarrow[-\frac{1}{2}R_3]{} \begin{pmatrix} 1 & -3 & 0 & | & -12 \\ 0 & 1 & 3 & | & 16 \\ 0 & 0 & 1 & | & 11 \end{pmatrix}$$

In the case that there is a unique solution, that is when the planes described by the original equations intersect in a unique point, we will always arrive at a situation in which ones are in the diagonal and zeros are below. In this form we can now read off the solution by reinserting the variables and the plus and equals signs. We get a new system of equations which has the same solutions as the first.

$$\begin{Bmatrix} x - 3y = & -12 \\ y + 3z = & 16 \\ z = & 11 \end{Bmatrix}$$

Substituting the value of $z = 11$ into the second equation, we have $y = -17$, and substituting this value of y into the first equation gives, $x = -63$.

1.5 Systems with infinitely many solutions - with no solutions

Given a system of equations we have seen that sometimes the planes they describe do not intersect in a unique point, and in some cases they may intersect in a common line in which case the x, y, and z coordinates of any point on the line form a solution to the system of equations. Other times, as in figures 6, 8, and 9, there are no points common to all three planes and the associated system of equations has no solutions. Such a system of equations is said to be *inconsistent*.

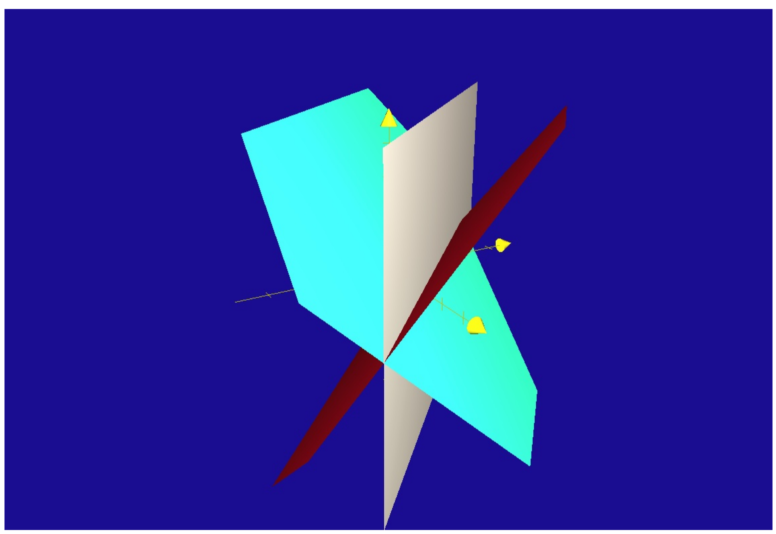

Figure 12: Three planes $x + y + z = 1, \quad z + y - z = 1$ and $x + y = -1$ intersecting in the line $y = -x + 1$ in the $z = 0$ plane

Example 4 Infinitely many solutions

Consider the system of equation

$$\left\{\begin{array}{r} x+y+z= \quad 1 \\ x+y-z= \quad 1 \\ x+y= \ -1 \end{array}\right\}$$

With corresponding matrix

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & 1 & -1 & | & 1 \\ 1 & 1 & 0 & | & 1 \end{pmatrix}$$

Gaussian elimination gives the following sequence of matrices

$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 1 & 1 & -1 & | & 1 \\ 1 & 1 & 0 & | & 1 \end{pmatrix} \xrightarrow[-1R_1+R_2 \ \& \ -1R_1+R_3]{} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & -2 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix} \xrightarrow[-\frac{1}{2}R_2]{} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & -1 & | & 0 \end{pmatrix}$$

$$\xrightarrow[-1R_2+R_3]{} \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Rewriting this last matrix as a system of equations we have

$$\left\{\begin{array}{r} x+y+z= \ 1 \\ 0x+0y+z= \ 0 \\ 0x+0y+0z= \ 0 \end{array}\right\}$$

So we know that $z = 0$ and substituting into the first equation we have $x + y = 1$ or $y = -x + 1$ which in the $z = 0$ is precisely the equation of the line of intersection of the three planes determined by the original equations. See figure 12

Example 5 No solutions

Given the system of equations

$$\left\{\begin{array}{r} x+2y= \ 1 \\ -2x-3y+z= \ 1 \\ 3x+5y-z= \ 2 \end{array}\right\}$$

Gaussian elimination gives the following sequence of matrices

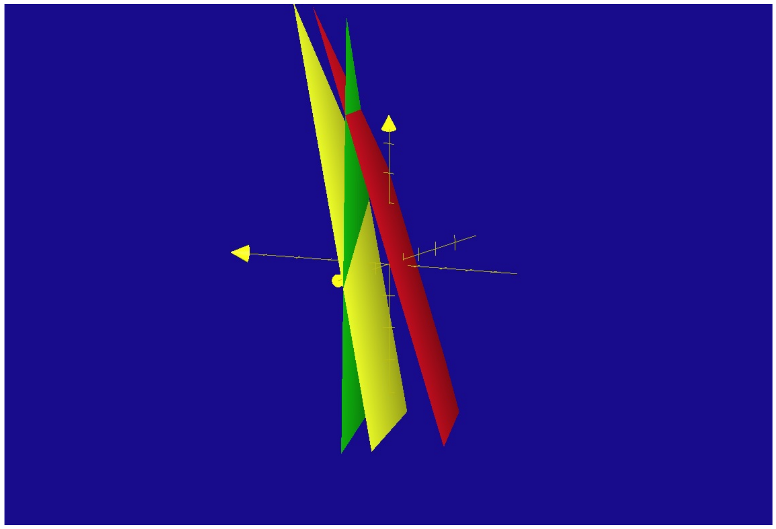

Figure 13: Three planes $x + 2y = 1, \quad -2x - 3y + z = 1$ and $3x + 5y - z = 2$ intersecting in pairs but with no common points of intersection. Therefore no solutions to system of equations

$$\begin{pmatrix} 1 & 2 & 0 & | & 1 \\ -2 & -3 & 1 & | & 1 \\ 3 & 5 & -1 & | & 2 \end{pmatrix} \xrightarrow[2R_1+R_2 \,\&\, -3R_1+R_3]{} \begin{pmatrix} 1 & 2 & 0 & | & 1 \\ 0 & 1 & 1 & | & 3 \\ 0 & -1 & -1 & | & -1 \end{pmatrix} \xrightarrow[R_2+R_3]{} \begin{pmatrix} 1 & 2 & 0 & | & 1 \\ 0 & 1 & 1 & | & 3 \\ 0 & 0 & 0 & | & 2 \end{pmatrix}$$

If we were to rewrite this as a system of equations, the last row would tell us that $0 = 1$ which is nonsense. The conclusion is that the system has no solutions and is inconsistent.

2 Counting

Many elementary questions involve counting tasks which if you have not been exposed to them can confusing. At Luigi's Pizza they have 10 different toppings other than the basic tomato sauce and cheese. How many different choices of four toppings are there. assuming that one of the toppings is mushrooms? Suppose the problem is complicated with the choice of 3 styles of pizza crust thick, regular and whole wheat. How many possible choices of 4 toppings with one mushrooms are there then? Fortunately there are basic principles that we can use as guides to a solution.

2.1 Basic counting principle

Lets consider the famous license plate problem, a variant of which goes like this:

In less than a minute calculate the number of Ontario license plate with letters and numerals in the standard format of four letters followed by 3 numerals, 0,1,2,3,4,5,6,7,8, or 9.

The answer is 456,976,000, and the solution rests on a basic principle. The solution goes like this

Solution: Examining in turn each of the 7 positions available for placing either a letter or one of the numerals, starting at the left

- there are 26 possibilities for assigning the first letter
- for each of the 26 possibilities for the first position there are 26 for the second for a total of $26 \times 26 = 26^2 = 676$ possibilities for assigning the first 2 letters
- for each of the 676 there are 26 ways to assign the third letter for a total of $26^3 = 17,576$ possibilities for assigning the first 3 letters
- for each of the $17,576$ possibilities there are 26 ways to assign the fourth letter for a total of $26^4 = 456,976$ ways to assign the four letters.
- there are 1,000 ways to assign the ten numerals $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ so for each of the 456, 976 ways of assigning the 26 letters to the first 4 positions there are 1,000 ways of assigning the subsequent 3 numerals. Thus there is a total of 456,976,000 license plate numbers in the specified format.

■

The principle basic principle behind this can probably be guessed. It is as as follows.

Principle 6 *:* **Counting principle**

Given a list of length n, if the first position of the list can be filled in m_1 ways, the second in m_2 ways, and so on with the n^{th} filled in m_n ways. Then the total number of ways of completing the list is $m_1 m_2 m_3 \cdots m_n$.

Example 7 *How many arrangements of 5 letters are there assuming that the first is a consonant and the second a vowel?*

Solution: There are 21 choices for the first position, 5 for the second and 26 for the third, fourth and fifth for a total of $21 \times 5 \times 26 \times 26 = 1,845,480$. ■

2.2 Permutations

An arrangement of a fixed number number of objects into a list is a permutation. In a given situation the task is to determine the number of possible permutations. For instance, how many permutations are there of the six letters A, B, C, D, E, F. In other words, in how many ways can these six letters be arranged into a list without repetition? The solution is obtained from the counting principle. There are 6 choices for the first position, and since one of the letters has now been used up there are 5 choices for the second, 4 for the third, and so on, with 1 choice for the last position. The total is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$. The principle involved is the following

Principle 8 *:* **Permutation**

The number of ways that n objects can be arranged to form a list or lined up is $n! = n(n-1)(n-2)\cdots(3)(2)(1)$

There are n ways of placing the first object because there are n elements to choose from. Having placed one there are then $n-1$ objects left and $n-1$ ways to place the second object, and so on. Note that the terminology $n!$ is shorthand for the words n *factorial* and is used to denote the product of the first n positive integers.

Now suppose there are fewer places in the list than objects. For instance consider the situation in which a class of 25 elementary school children are asked to form a line of 10 students. How many ways can this be done? The solution depends on the basic counting principle. There are 25 to fill the first position, 24 for the second because one of the 25 students has already been placed, 23 ways to fill the 3 position and so on, with 16 ways to fill the 10^{th} and last position. Using factorial notation

$$(25)(24)(23)(22)(21)(20)(19)(18)(17)((16) = \frac{25!}{15!} = \frac{25!}{(25-10)!}$$

The principle here is stated in terms of forming a lists of length r from a collection of n objects or in other words permutations of n objects taken r at a time.

Principle 9 *:* **Permutations of n objects taken r at a time**

Given a collection o n objects a list consisting of r of them can be formed in

$$P(n,r) = \frac{n!}{(n-r)!}$$

different ways

The counting principle tells us that the number of ways to arrange n objects in a list of length r is

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$$

and this is the same as $P(n,r) = \dfrac{n!}{(n-r)!}$ after seeing that the denominator $(n-r)!$ cancels with the same term in the numerator.

$$\begin{aligned} P(n,r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= n(n-1)(n-2)\cdots(n-r+1)\cdot\frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)} \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

Lets consider one further complication. Suppose we are given a collection of n objects in which some can not be distinguished from one another. For example suppose we have a collection of ten letters and that four of them happen to the letter A. The question is - in how many distinguishable ways may the letters be lined up. Here is how the reasoning goes. Let x denote the number of distinguishable arrangements of the eight letters. For each such arrangement there are a further 4! arrangements obtained by rearranging the As. Thus $4!x$ is the total number of arrangements which we know equals 10! - so $10! = 4!x$, and the number of distinguishable arrangements is $x = \frac{10!}{4!}$

The ideas presented in the example above can be generalized to more than one group of indistinguishable objects. Consider the following example.

Example 10 *In how many ways can four pennies, three dimes, four quarters, an 5 loonies be arranged in a row?*

Solution: There are a total of 16 coins. We know that if the dimes, quarters, and loonies had some distinguishing marks but not the pennies, then there would be $\frac{16!}{4!}$ distinguishable permutations. Now lets take away the distinguishing marks on the collection of dimes. Our previous discussion then says that there would now be $\frac{\frac{16!}{4!}}{3!} = \frac{16!}{4!3!}$ distinguishable permutations. Next remove the distinguishing marks from first the quarters and then the loonies. Following the same logic there are then $\frac{16!}{3!4!4!5!}$ distinguishable permutations.

■

These ideas can be formalized as follows

Principle 11 *:* **Distinguishable permutations**

Given a collection of n objects suppose that $n = k_1 + k_2 + \cdots k_r$ and that the objects are of different types, with k_1 of type 1, k_2 of type 2, $\cdots$ and k_r of type r, then the number of distinguishable permutations is

$$\frac{n!}{k_1!k_2!k_3!\cdots k_r!}$$

2.3 Combinations

When we spoke of permutations, we spoke of lists and arrangements . In other words we always had an ordering of the objects in mind. In studying combinations, we assume no ordering. The problem is given a collection of n objects, how many ways are there to choose r of them where r is some number less than n? Each such choice is called a combination. To answer the question we use the same logic as before. Let x denote the number of combinations of n objects taken r at a time. For each such combination there are $r!$ permutations (rearrangements). There are x such combinations so thatn $r!x$ is then the total number of permutations of the n objects taken r at a time. In other words

$$\frac{n!}{(n-r)!} = r!x$$

so that

$$x = \frac{n!}{(n-r)!r!}$$

Principle 12 *:* **Number of combinations of r objects chosen from n**

The number of combinations of n objects taken r at a time is n choose r is

$$C(n, r) = \frac{n!}{(n-r)!r!}$$

Example 13 *In a class of 25 students, how many ways can a study group of 5 students be formed?*

Solution: Here $n = 25$ and $r = 5$. The answer is $C(25, 5) = \frac{25!}{5!20!} = 53,130$ ■

These are the basics. There are problems that involve both a calculation of both permutations and combinations. Here is an example.

Example 14 *A condominium management committee consisted of a chair person, a vice chair, a secretary and 5 others chosen at large from the 50 residents. How many ways can this committee be chosen?*

Solution: The chair, vice chair, and secretary can be selected in $P(50, 3) = (50)(49)(48) = 117,600$ ways. And for each such selection there are now 47 residents from whom to choose the 5 others - that is 47 choose 5 or

$$C(47, 5) = \frac{47!}{42! \cdot 5!} = \frac{(47)(46)(45)(44)(43)}{5!} = 533,939.$$

The total number of committees is then the product $P(50, 3)C(47, 5) = 117,600 \times 1,533,939$, a rather large number. ■