1 Polynomial functions

Polynomial functions are the most basic. They are well understood and they are easy to calculate with. Before the advent of computers - not so long ago - all calculations were done essentially by hand. As a result enormous amounts of thought and energy went into the task of figuring out how to approximate other functions with polynomials. Trigonometric and exponential/logarithmic functions occur very naturally, but how does one calculate with them? We approximate with polynomials. Now all of this is hidden within our calculator or computer but the techniques are no different than those discovered hundreds of years ago. The strength of this approach lies with the fact that any function can be approximated to any desired degree of accuracy by some polynomial function.

After all this, what is a polynomial function. A polynomial function is one which accepts any real number x as input and through a process of multiplication and addition output is produced. The multiplication may or may not involve a collection of constants. The following are examples

$f(x) = x + 1$,
$f(x) = 2x^3 + 3$,
$f(x) = -\pi x^5 + 3x^2 + 4x + 1$

The first is said to be a linear function because its graph is a line. The second is said to be a cubic polynomial and the third is a quintic polynomial. More generally given $n + 1$ constants, $a_0, a_1, a_2, \cdots, a_n$, the general n^{th} degree polynomial function is written

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

In more detail there are the following definitions. Notice that we make the distinction between a *polynomial* as simply an abstract expression and a *polynomial function* where the polynomial is used as the rule of calculation for the function.

Definition 1 Polynomial of degree n

Given a natural number n and real numbers $a_0, a_1, a_2, \cdots a_n$ the associated polynomial of degree $n$is the expression
$$a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n.$$

The numbers $a_0, a_1, a_2, \cdots a_n$ are called the coefficients *of the polynomial.*

Definition 2 *Polynomial function*

Given a polynomial of degree n, $a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$, the associated polynomial function is the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n$$

Here is an interesting fact. We know that any two points determine a unique line. Well its also true that any 3 points determine a unique quadratic function, any 4 a unique cubic, and any $n - 1$ a unique polynomial function of degree n. And - this is the fundamental reason why in general the graph of a quadratic will have one bump, a cubic two bumps, a quartic 3 bumps, and a degree n polynomial function will have $n - 1$ bumps. We can see this illustrated with the following examples.
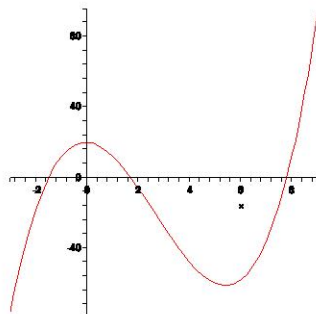

Figure 1: the cubic $f(x) = x^3 - 8x^2 - x + 20$ with 2 bumps
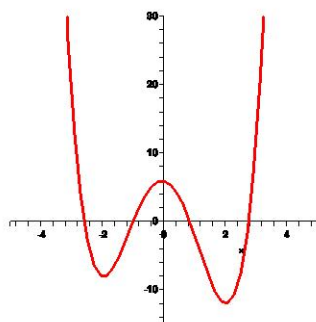

Figure 2: graph of quartic $f(x) = x^4 - 8x^2 - x + 6$ with 3 bumps

1.1   Zeros, Roots and Factors

Given a polynomial function f a zero or in other words a root of f is a number x with the property that $f(x) = 0$, and graphically a zero is a point at which the graph of the function crosses the x axis. Now there is an interesting fact. If $x = a$ is a zero of f so that $f(x) = 0$, then it follows that $(x - a)$ is a factor of the polynomial. This is easily shown to

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be true. If you take any polynomial of say degree n and divide it by a linear polynomial, what you end up with is a polynomial of degree one less and a constant remainder. Just try it. Suppose we have the polynomial function $f(x) = x^3 - 6x^2 + 11x - 6$, where it so happens that $x = 1$,
$x = 2$ and $x = 3$ are zeros of f. Lets try dividing by $(x - 4)$t If we do the division we get

$$\frac{x^3 - 6x^2 + 11x - 6}{x - 4} = x^2 - 2x + 3 + \frac{6}{x - 4}$$

or multiplying through by $(x - 4)$ we have

$$x^3 - 6x^2 + 11x - 6 = (x - 4)(x^2 - 2x + 3) + 6$$

On the other hand if we divide by x minus a root, say $(x - 2$, we see there is no remainder and that

$$x^3 - 6x^2 + 11x - 6 = (x - 2)(x^2 - 4x + 3)$$

In the general case, if we have a degree n polynomial f and a linear polynomial $(x - b)$, then there is a degree $n - 1$ polynomial g and a number r such that

$$f(x) = g(x)(x - b) + r$$

Knowing this, for $x = b$ to be a zero, it must be true that $r = 0$, so that consequently $(x - b)$ is a factor. Conversely if $(x - b)$ is a factor, then $f(x) = g(x)(x - b)$, and it follows immediately that $x = b$ is a zero. All of this can be stated as a theorem.

Theorem 3 *If f is a polynomial of degree n, Then $x = a$ is a zero(root) of f implies that $(x - a)$ is a factor of f and conversely if $(x - a)$ is a factor of f, then $x = a$ is a zero of f. Phrased more succinctly, $x = a$ is a zero(root) of f if and only if $(x - a)$ is a factor of f.*

1.2   Rational functions

A rational function is a quotient of two polynomial functions, for instance

$$f(x) = \frac{x^2 - 1}{x - 2}$$

describes a rational function. Rational functions are defined for all real numbers which do not give a zero for the function in the denominator. In our example, the function f is not

defined at $x = 2$ because for this value the denominator is zero. We see however that for x close to 2 but less than 2, say $x = 1.999$, the denominator has value $1.999 - 2.0 = -0.001$ and the value of the function is $f(1.999) = \frac{1.999^2 - 1}{0.001} = -2996.001$ which is rather large as a negative number, and in fact, as x gets arbitrarily close to 2 but still is less than 2 the value of $f(x)$ becomes a near infinitely small negative number. Now for $x > 2$, the analysis is the same, but $f(x)$ now becomes a near infinitely large positive number. The graph is shown below. Observe that the vertical red line is not part of the graph, but is included to show that the graph of the function becomes arbitrarily close to this line. When such a thing happens, we say that the graph is *asymptotic* to the vertical line.
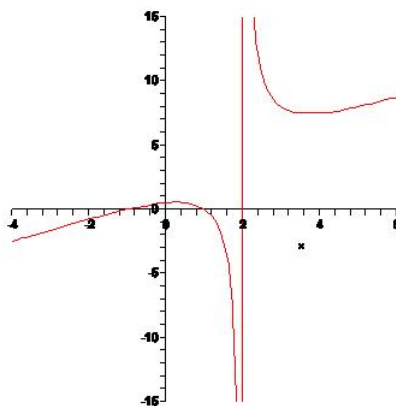

Figure 3: graph of rational $f(x) = \frac{x^2 - 1}{x - 2}$

1.3 Finding zeros of functions

Given a subset A of real numbers and a function $f : A \to \mathbb{R}$, a common practical task is that of finding for which values of x, $f(x) = 0$. In terms of the graph of the function, this tells us where the graph crosses the x axis. If f is a polynomial with integer coefficients, there are a host of methods that can be used to help determine what rational numbers are zeros of f. In the case of a quadratic function, $f(x) = ax^2 + bx + c$, there is a nice formula, which we all know, that tells us all the zeros, rational, irrational, or complex. In the case of degree 3 or degree 4 polynomial functions, there are also increasingly complex formula for the zeros.

One of the outstanding achievements was the discovery by a 21 year old French university student, Evariste Galois, in 1832, is that there is no formula using normal algebraic operations that will tell us all the zeros of any polynomial whose degree is a prime greater

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than or equal to 5. Galois at his young age tragically died from wounds sustained in a duel on May 31, 1832. Legend has it that he was romantically defending the honor of a young woman with whom he had fallen in love.

For functions other than polynomials, trigonometric, logarithmic, or exponential (we'll get to this stuff eventually), there are also no universal methods for determining zeros.
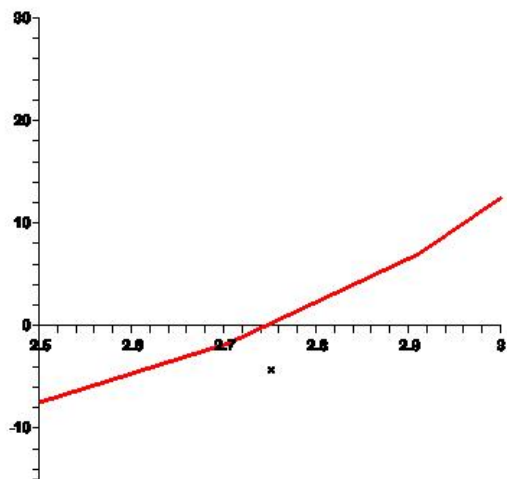

Figure 4: zooming in on a zero of $f(x) = x^4 - 8x^2 - x + 6$

So lacking any universal method for finding zeros of functions, it is most often necessary to use approximation methods. One of these, Newton's Method which works for any function will be expanded on later in the context of a discussion of calculus. A much more simple method but with somewhat less accuracy is the graphical method in which one uses a computer graphing programme and simply zooms in on where the graph crosses the x axis. For instance examining the graph in figure **??** it appears that on the far right that the function crosses the x axis somewhere between 2.5 and 3. In figure **??** using the zooming capacity of the Maple software, it appears more accurately that the graph crosses somewhere between 2.73 and 2.75. Zooming further one can obtain a yet more accurate approximation.

Although it is not necessarily easy to find the zeros of even a polynomial function we are aided by a result called the *Fundamental Theorem of Algebra.* This states that any polynomial can be factored completely into linear factors and that the associated roots (zeros) may be real or purely complex .

Theorem 4 Fundamental Theorem of Algebra

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If $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is a polynomial function, there exist numbers $r_1, r_2, r_3, \cdots r_n$ real or purely complex such that

$$f(x) = (x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n)$$

and that if any of the roots $r_1, r_2, \cdots, r_n$ say r_i is purely complex of the form $a + bi$, then $a - bi$ is also a root.

In other words purely complex roots occur in pairs, one the complex conjugate of the other. Although, this does not provide a method for finding zeros, one at least knows it is possible, and that should it be possible to find one root say $x = r$, then one can divide by $(x - a)$ obtaining a polynomial of lower degree, and maybe you will end by being lucky and find a root of this new polynomial. If so you divide again, and so on. Also if you should stumble on a complex root, then you immediately know that its conjugate is also a complex root.

Consider the following example.

Example: If $f(x) = x^3 - x^2 + x - 1$, it is easy to see that $x = 1$ is a root. Just substitute $x = 1$ into the definition of f. That means that $(x - 1)$ is a factor, and dividing by $(x - 1)$ gives
$$x^3 - x^2 + x - 1 = (x - 1)(x^2 + 1) = (x - 1)(x - i)(x + i).$$

1.4 Moving, Reflecting, Flattening, Expanding, Stretching, Compressing

Given the graph $\mathcal{G}$ of either a function $f : A \to \mathbb{R}$ defined on a subset A of $\mathbb{R}$ or or some equation of two variables, suppose the graph is altered by moving the graph, squeezing or expanding the graph, or reflecting the graph about one of the axes . If $\mathcal{G}'$ is the altered graph,. how does one change the description of the function or the equation so that the resulting graph, call it $\tilde{\mathcal{G}}$, is the same as $\mathcal{G}'$? This is not so hard. By considering the graph of a function f as the graph of the two variable equation $y = f(x)$, we consider only the case when we wish to move the graph of an equation.

1.4.1 Moving the graph horizontally or vertically

Consider the graph of a function f as the graph of the equation of two variables $y = f(x)$. Lets begin by looking at moving a graph horizontally, and lets focus on a simple example - say $f(x) = y = x$ whose graph $\mathcal{G}$ is the diagonal line of Figure 5 . Suppose $\mathcal{G}$ is moved one unit to the right getting a new graph $\mathcal{G}'$. Now, if (x, y) is a point on $\mathcal{G}'$, then $(x - 1, y)$ must be on $\mathcal{G}$ - we have simply moved a point back to where it came from. This must mean

however that $(x-1, y)$ is on the diagonal line - that is: $y = x - 1$. Hence, for a point to be on moved graph it must satisfy $y = x - 1$. Everything we have said holds equally well for an arbitrary function or equation and its graph. Thus, we have the following result.

Result: To move a graph of an equation a units to the right , where a is any positive real number, the equation for the new graph is obtained by subtracting a from every occurrence of the x variable. To move the graph a units to the left, the equation for the new graph is obtained by adding a to each occurrence of the x variable
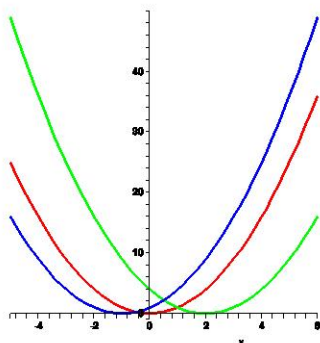

Figure 5:

Thus to move to the right you subtract and to move to the left you add. Figure **??** shows an example. The blue curve is the graph of $y = (x + 1)^2$, the red the graph of $y = x^2$, and the green the graph of $y = (x - 2)^2$

An entirely similar argument shows that having moved a graph vertically up, one must add the amount to each occurrence of the y variable to obtain a new equation for the transformed graph and to move it vertically down one subtracts the amount . So for the equation $y = x^2$, if we wish to move the graph up say 3 units, the equation which describes the graph is $y - 3 = x^2$, which is then written as $y = x^2 + 3$. If we were to move, the graph downwards 3 units we must add to the y variable getting the equation $y + 3 = x^2$ or $y = x^2 - 3$

Result: the graph of an equation has been moved b units upwards , where b is any positive real number, the equation describing the new graph is obtained by subtracting b from every occurrence of the y variable. If the graph has been moved downwards b units, the equation is obtained by adding b to every occurrence of the y variable.

1.4.2   Reflecting a graph about the x or y axis

To reflect a graph about the y axis we argue in the same way. Running through it again - given a graph $\mathcal{G}$ of some equation, let $\mathcal{G}'$ be the result of reflecting $\mathcal{G}$ about the y axis. How does one change the equation so that its graph is the same as $\mathcal{G}'$? Again we begin by looking at a point (x, y) on the transformed graph $\mathcal{G}'$ Moving (x, y) back to its position on the first graph involves putting a negative sign in front of x. We get $(-x, y)$. Since $(-x, y)$ is now on the first graph, it must satisfy the original equation. So we summarize as follows:

Result: The equation for a graph that has been reflected about the y axis is obtained by putting a negative sign in front of each occurrence of the x variable.

In the same way we can obtain an equation for the graph when reflected about the x axis.

Result: The equation for a graph that has reflected about the x axis is obtained by putting a negative sign in front of each occurrence of the y variable.

Following are some examples. In figure **??** the graph of $y = (x - 2)^2 - 2$ is reflected about the y axis, and the resulting graph is described as the graph of $y = (-x - 2)^2 - 2$. In figure **??** the graph of the same function is reflected about the x axis and the equation describing the graph is obtained by placing a negative sign in front of the y variable getting $-y = (x - 1)^2 - 2$. This is written more conventionally as $y = -(x - 1)^2 + 2$.
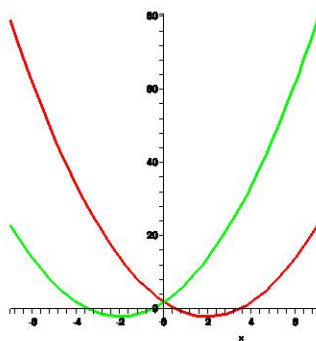

Figure 6: reflection about the y axis

1.4.3   Expanding or flattening

Consider the functions $f(x) = x^2$ and compare the graph to that of $f(x) = 2x^2$ and to that of $h(x) = \frac{1}{3}x^2$. The graphs of each are shown in figure **??**. The central one in red is the
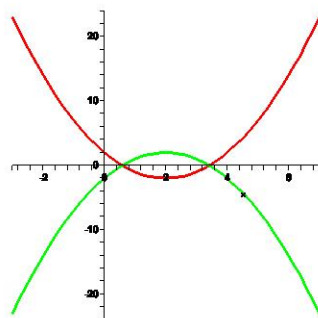
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Figure 7: reflection about the x axis

graph of $f(x) = x^2$, the green one which is expanded vertically by a factor of $\frac{1}{2}$ and is the graph of $f(x) = 2x^2$, and the blue one which is flattened by the factor of 3 and is the graph of $h(x) = (\frac{x}{3})^2$.
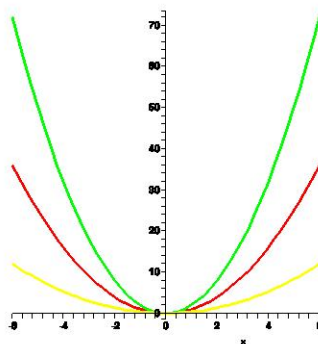

Figure 8: expandiong and flattening the red parabola $y = x^2$

Result: *To expand a graph of a function vertically multiply the expression defining the function by a constant greater than 1. To flatten the graph multiply by a constant between zero and one.*

Expressing the graph as the graph of the equation $y = f(x)$, we have said to expanded by say a factor of two, we need to consider the equation $y = 2f(x, y)$. This is the same thing as writing $\frac{y}{2} = f(x, y)$. Now, we can say:

To expand vertically by some factor greater than one, we need to divide the y variable by the same constant Similarly for flattening, divide the y variable by a constant between zero and one.
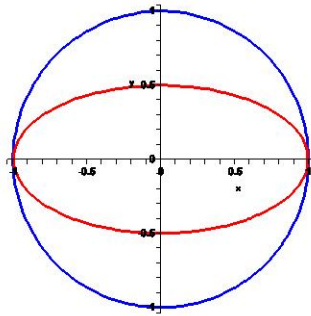
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Figure 9: an ellipse in blue with equation $(x)^2 + (2y)^2 = 1$

This technique of expanding or flattening the shape by multiplying the function definition by a constant works reliably for functions. This process can be rephrased in terms of multiplication of the y variable. Notice that in the example above the three graphs are also the graph of the equations ,

$y = x^2, y = 2x^2$, and $y = (\frac{x}{3})^2$, and that the last two are the same as: $\frac{1}{2}y = x^2$ and $3y = \frac{y}{\frac{1}{3}} = x^2$.

Looking at things this way, we have divided the y variable by 2 to expand and we have divided the y variable by $\frac{1}{3}$ to flatten. See figure **??**.

The same thing can be done with certain equations such as that of a circle

In figure **??** there are two graphs. The blue one is that of the equation $x^2 + y^2 = 1$ and the red is that of the ellipse $x^2 + (2y)^2 = 1$, in which the y variable has been multiplied by 2 or equivalently divided by$\frac{1}{2}$.

1.4.4 Stretching and compressing

The question that now naturally arises is, what happens if we adjust the x variable by dividing or multiplying? Lets look at some examples. The best one to illustrate the differences is the sine function, which we will speak much more of later. The graph of the sine undulates between -1 and 1 running from $-\infty$ to ∞, see figure **??**.

If we multiply the x variable by 2 or equivalently divide it by $\frac{1}{2}$ the graph undulates twice as fast. In figure **??** the red line is the graph of $y = sinx$ and the green is that of $y = sin(2x)$, which can be viewed as a horizontally compressed version of the red line. On the other hand if we divide the x variable by a number greater than 1 - say 2, then the

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graph is stretched in the horizontal direction. In figure **??** green line undulates twice as slowly.

This same behavior is seen with all graphs - although for some graphs - especially those that are symmetric about the y axis, manipulating the x variable often has a similar effect if one were to do the reverse operation with the y variable - that is if you multiply the x variable, try dividing the y variable. This is best seen through examples using some computer graphing software or a graphing calculator.
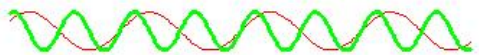
Figure 10: Graph of $y = sin x$

Figure 11: Graph of $y = sin(x)$ and $y = sin(2x)$ - the latter in green

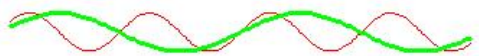

Figure 12: Graph of $y = sin(x)$ and $y = sin(\frac{x}{2})$ - the latter in green