Appendix B

Uncertainty Calculations using Differentials

If x, y etc., are measured quantities how does one calculate the resultant uncertainty in p where

(i) $p = x + y$

(ii) $p = x - y$

(iii) $p = xy$

(iv) $p = x/y$

and combinations of all the above?

Let the absolute uncertainties be interpreted as differentials dx, dy, etc

Consider (i) $\quad p = x + y$

Taking differentials $dp = dx + dy$

and considering (ii) $\quad p = x - y$

Taking differentials $dp = dx - dy$

The resultant uncertainty which we estimate is to be a maximum (upper) and minimum (lower) bound. Since the uncertainties are always $\pm dx$; $\pm dy$, etc., we must, therefore, choose the sign which will give the greatest uncertainty. This, in effect, means that after the differentials have been taken, as a last step before substituting numbers, we should change all - signs to + signs.

Therefore, for addition and subtraction $dp = dx + dy$ i.e. the final uncertainty is the sum of absolute uncertainties.

Consider

$$\begin{aligned} \text{(iii)} \quad p &= xy \\ \text{Taking differentials : } \quad dp &= xdy + ydx \end{aligned}$$

This would be complicated to compute, but may be simplified by dividing (2) by (1).

$$\frac{dp}{p} = \frac{xdy + ydx}{xy} = \frac{dy}{y} + \frac{dx}{x} \tag{B.1}$$

Note that $dp/p, dx/x, dy/y$ are just the relative uncertainties in p, x, y.

Multiply through equation (3) by 100 to convert to a percentage uncertainty

$$\frac{dp}{p} \times 100 = \frac{dx}{x} \times 100 + \frac{dy}{y} \times 100$$

For division: Consider

$$\begin{aligned} \text{(iv)} \quad p &= x/y \\ \text{Taking differentials : } \quad dp = \frac{ydx - xdy}{y^2} \end{aligned}$$

This also seems complicated to compute but can be simplified by dividing equation (5) by equation (4):

$$\frac{dp}{p} = \frac{\frac{ydx - xdy}{y^2}}{\frac{x}{y}} = \frac{ydx - xdy}{y^2 x} y$$

$$\frac{dp}{p} = \frac{dx}{x} - \frac{dy}{y}$$

By the same reasoning as in (ii) above, we will now change the -sign to + sign.

$$\frac{dp}{p} = \frac{dx}{x} + \frac{dy}{y}$$

which is the same as (3)above.

Therefore, for multiplication and division, the final uncertainty, expressed as a percentage, is the sum of percentage uncertainties of the components.

Note that if one of the factors is taken to a power e.g. $p = y^a$ then by differentials

$$\begin{aligned} dp &= ay^{a-1}dy \\ \frac{dp}{p} &= \frac{ay^{a-1}dy}{y^a} = \frac{ady}{y} \end{aligned}$$

However, if $p = 2xy$ then $dp = 2(xdy + ydx)$ so

$$\frac{dp}{p} = \frac{2(xdy + ydx)}{2xy} = \frac{dy}{y} + \frac{dx}{x}$$

When addition, subtraction, multiplication and division are all involved in one formula, things become more complicated. These may be treated by the differential process or by using the rules on one part of the function at a time.

Examples The dimensions of a room are:

$$\begin{aligned} \text{length (l)}: &\quad (3.04 \pm .02) \text{ m} \\ \text{width (w)}: &\quad (3.66 \pm .02) \text{ m} \\ \text{height (h)}: &\quad (2.74 \pm .02) \text{ m} \end{aligned}$$

What are the resultant uncertainties in

(a) the perimeter of the floor?

Perimeter $P = 2\times$ length + 2 $\times$ width $= 2l + 2w$ = 2(3.04) + 2(3.66) = 13.40 m.

Uncertainty in perimeter is $\delta P = 2\delta l + 2\delta w = 2(0.02) + 2(0.02) = 0.08$ m.

So the perimeter of the floor is therefore (13.40 ± 0.08) m.

(b) the floor area?

Area A_{floor} = length $\times$ width $= l \times w$ = (3.04 m)$\times$(3.66 m) = 11.13 m^2

Uncertainty in Area is :

$$\begin{aligned} \frac{\delta A_{\text{floor}}}{A_{\text{floor}}} &= \frac{dl}{l} + \frac{dw}{w} \\ \delta A_{\text{floor}} &= \left(\frac{dl}{l} + \frac{dw}{w}\right) A_{\text{floor}} \\ &= \left(\frac{0.02}{3.04} + \frac{0.02}{3.66}\right) 11.13 \\ &= 0.13 \text{ m}^2 \end{aligned}$$

So the area of the floor is (11.13 ± 0.13) m^2

(c) the wall area?

$$\begin{aligned} A_{\text{walls}} &= 2(\text{length} \times \text{height}) + 2(\text{width} \times \text{height}) \\ &= 2(3.04)(2.74) + 2(3.66)(2.74) \\ &= 36.71 \text{ m}^2 \end{aligned}$$

and

$$\begin{aligned}
\delta A_{\text{walls}} &= \delta(2l \times h) + \delta(2w \times h) \\
\delta(2l \times h) &= \left(\frac{\delta l}{l} + \frac{\delta h}{h}\right) 2lh \\
\delta(2l \times h) &= \left(\frac{0.02}{3.04} + \frac{0.02}{2.74}\right) 2(3.04)(2.74) \\
&= 0.23 \text{ m}^2 \\
\delta(2w \times h) &= \left(\frac{\delta w}{w} + \frac{\delta h}{h}\right) 2lh \\
\delta(2l \times h) &= \left(\frac{0.02}{3.04} + \frac{0.02}{2.74}\right) 2(3.04)(2.74) \\
&= 0.26 \text{ m}^2 \\
\delta A_{\text{walls}} &= (0.23 + 0.26) \text{ m}^2 \\
&= 0.49 \text{ m}^2 \;=\; 0.5 \text{ m}^2
\end{aligned}$$

So the area of the walls is (36.7 ± 0.5) m^2.