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Optimal asset allocation in life annuities: a note

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Abstract

In this note, we derive the optimal utility-maximizing asset allocation between a risky and risk-free asset within a *variable annuity* (VA) contract, which is a US-based savings and decumulation investment product. We are interested in the interaction between financial risk, mortality risk and consumption, towards the end of the life cycle. Our main result is that for constant relative risk aversion (CRRA) preferences and geometric Brownian motion (GBM) dynamics, the optimal asset allocation during the annuity decumulation (payout) phase is identical to the accumulation (savings) phase, which is the classical Merton [J. Econ. Theory 3 (1971) 373] solution. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction and motivation

The purpose of this brief note is to derive the optimal asset allocation in a variable annuity (VA) contract in the payout phase, which normally coincides with retirement. Our results are obtained in a multi-period framework in which the individual maximizes his/her expected utility of consumption. We label this particular allocation the retirement asset mix.

In the United States a VA contract is a tax-sheltered retirement savings plan to which after-tax contributions are added on a periodic basis. These contributions are then allocated amongst a variety of investment sub-accounts whose gains will not be taxed until the annuity is surrendered or cashed. There is currently US$ 1 trillion invested in these annuity accounts. Contributions in 1999 alone amounted to approximately US$ 100 billion. [1]

At retirement, the accumulation phase of the VA is terminated, and the payout phase begins. The plan holder has two choices: (i) he/she can simply withdraw the funds at once and face a fairly large tax liability; or (ii) all or part of the funds in the account can be annuitized. If the annuitization option is chosen, the holder has the ability to select an asset mix upon which the annuity payments will be based. Naturally, the optimal annuitization policy and the ensuing retirement asset mix depend on the individual's attitude towards mortality and financial market risks.

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[1] Source: Life Insurance Market Research Association (LIMRA) and Moodys Investor Services, October 2000. See Daily (1994) or Williamson (1999) for an excellent practitioner-oriented description of this *relatively* novel product.

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We contrast the optimal retirement asset mix with its pre-retirement (i.e., accumulation phase) counterpart. Our goal is to determine whether the annuitization option, which functions as longevity insurance, has any effect on the individual's optimal asset mix. Specifically, we are interested in the exposure to risky assets (i.e., the equity funds) pre- and post-retirement.

Recent literature in the area, such as Jagannathan and Kocherlakota (1996), Bodie et al. (1992), has examined the conditions under which the allocation to the risky asset would decrease as one approaches retirement. However, those analyses were not done in the presence of variable annuities. Now, since variable annuities have become very popular as a retirement saving vehicle, our research is relevant to many individuals' decision making. Our main result is that for constant relative risk aversion (CRRA) preferences and geometric Brownian motion (GBM) price dynamics, the optimal asset allocation during the annuity decumulation (payout) phase is identical to the classical Merton (1971) solution. This illustrates the robustness of the Merton (1971) optimum, even when it is applied to mortality-contingent claims. From a technical actuarial point of view, we contribute to the literature by developing a continuous-time model of the payout from a variable life annuity.

We must note, however, that our analysis focuses only on the asset allocation decisions, pre- and post-retirement, within the VA contract. We are not addressing the equally important question of what fraction, if any, of liquid assets should be invested in VA contracts during the accumulation phase. In addition, we will not address issues of optimal timing of annuitization within the context of variable annuities.

The remainder of the note is organized as follows. In the next section, we reproduce and review the well-known Merton's (1971) results on the optimal pre-retirement asset allocation. In Section 3, we discuss the lifelong risk-free and risky post-retirement income that the plan holder will receive if annuitization is chosen. Then, in Section 4, we present our results of the optimal post-retirement allocation in a VA contract. Section 5 concludes the note.

2. Pre-retirement asset allocation

The model and assumptions in this section follow closely the classic analysis by Merton (1971, 1993). We reproduce those results, simply to fix notation and terminology and for the purpose of comparison to the post-retirement behavior, which is our main contribution.

An individual starts a VA contract at time zero with an investment denoted by w. All liquid wealth is invested in the VA. The VA is maintained until time T (retirement), at which point it can be converted to an "immediate" annuity. In addition to w, further savings are exogeneously added to the VA account on a periodic basis. The new savings flow into the account—without any loss of generality—at a constant rate of $s\,\mathrm{d}t$, where $s \geq 0$. During the accumulation phase of the VA, all dividends, interest and realized capital gains are re-invested in the account, and all taxation is deferred until the payout phase begins at retirement. In the payout phase, all gains are taxed as ordinary income. In other words, there is no distinction between capital, dividends and interest.

Within the VA account, the investment is assumed to be partitioned between two sub-accounts. The first is a *risk-free* asset sub-account (such as a money market fund) and the second is a *risky* asset sub-account (such as a well-diversified equity index fund). The limited choices may seem restrictive in practice, considering the range of real-world products. However, the main asset allocation ideas are captured, without loss of generality, since the two choices can always be viewed within a separation-theorem context.

We denote by α_t the proportion of wealth allocated to the risky sub-account at any time t, and $(1 - \alpha_t)$ the proportion of wealth allocated to the risk-free sub-account at that time. We, quite realistically, disallow leverage and short-sales in the VA contract, which restricts α_t to be between 0 and 1; i.e., $0 \leq \alpha_t \leq 1$. Therefore, at any time $0 \leq t < T$, a nominal amount $W_t(1 - \alpha_t)$ is allocated to the risk-free asset, which earns a constant rate of return of r per annum. The remainder, $W_t\alpha_t$, is placed in the risky asset whose expected rate of return is μ per annum and whose volatility is σ per annum. The dynamics of the risky asset, and therefore wealth, are assumed to obey a GBM. The aggregate wealth process of the VA account will satisfy the following stochastic differential equation:

$$\mathrm{d}W_t = W_t[\alpha_t\mu + (1 - \alpha_t)r]\,\mathrm{d}t + s\,\mathrm{d}t + \alpha_t\sigma W_t\,\mathrm{d}B_t, \quad 0 \leq t < T, \quad W_0 = w, \tag{1}$$

where B_t is a standard Brownian motion. Among other things, Eq. (1) states that as the new savings are added to the VA account, they are partitioned between the two sub-accounts in the same ratio as the existing allocation. Note that we are assuming that all expenses charged to the VA are implicitly absorbed in the parameters μ and r. As such, μ and r are the instantaneous *net* rates of return.

To determine the optimal asset allocation during the pre-retirement stage of the VA, the individual maximizes his/her expected utility of wealth at retirement, T. The utility function in question is assumed to be of a CRRA type with a functional form: $U(x) := (1/(1-\gamma))x^{1-\gamma}$ with the restrictions that $\gamma > 0$, thus excluding any risk-loving behavior. Also, when $\gamma = 1$ we define $U(x) := \ln[x]$. The pre-retirement asset allocation objective function is therefore

$$\max_{\alpha_t} E\left[\frac{1}{1-\gamma} W_T^{1-\gamma}\right], \tag{2}$$

subject to the constraint in Eq. (1) and that $W_0 = w$. The fixed time horizon implies no possibility of death prior to the annuitization stage. This may seem optimistic in practice, but does not materially affect the main results. Also, some annuities contain implicit put options that guarantee a return of premium upon death. We ignore this relatively minor institutional complication in our preliminary analysis, and instead refer the interested readers to Milevsky and Posner (2001) for more information on the embedded options.

As mentioned earlier, the solution to the objective function (2) is well known from the standard texts such as Merton (1993) or Karatzas and Shreve (1992). The optimal allocation to the risky asset is equal to

$$\alpha^* = \min\left[\frac{\mu - r}{\gamma\sigma^2}, 1\right]. \tag{3}$$

As one would expect intuitively, the allocation to the risky asset is increasing in the risk premium (defined as $\mu - r$), decreasing in the volatility (σ) and decreasing in the coefficient of relative risk aversion (defined as γ). Also, given that the risk premium $\mu - r$ is always positive and $\gamma > 0$, we are essentially guaranteed that $\alpha^* \geq 0$.

3. Post-retirement income from annuities

To examine the optimal investment allocation in the payout phase of the VA, we assume that the individual will annuitize all the funds in his/her account and there are no other assets to consider. In other words, the alternative of liquidating the account and consuming his/her wealth in a discretionary manner is not chosen. A theoretical justification for this assumption is provided by Yaari (1965), or more recently by Strawczynski (1999), who showed that an individual with no utility of bequest will annuitize (or invest in actuarial notes for) all of their liquid assets. We note that this assumption may not be entirely consistent with the empirical evidence that a majority of variable annuities are not-annuitized. See, for example, Poterba and Wise (1996) or Sondergeld (1997) for a discussion of empirically observed annuitization behavior. However, it enables us to achieve our goals of studying the asset allocation *within* the life annuity account and making a meaningful comparison between pre- and post-retirement decisions.

In our framework, the individual will have to decide again at the start of the payout phase how much of the accumulated wealth within the VA contract, to allocate to the risk-free sub-account (which will produce fixed annuity payments), and how much to allocate to the risky sub-account (which will produce uncertain or VA payments). In other words, the decision has to be made between a consumption flow out of a “fixed immediate annuity” (FIA), a “variable immediate annuity” (VIA) or a suitable combination of them.

In practice, there are various degrees of asset allocation ‘freedom’ within the payout phase of a VA contract. At one end of the freedom spectrum, some products and companies do not allow any changes in asset allocation once the payout phase has begun. The annuitization thus imposes a buy-and-hold restriction that prohibits any shifts—even to rebalance—from variable to fixed or vice versa. At the other end of the spectrum, are products, albeit rare, that allow complete and instantaneous mobility between the two types of annuity flows. With these, the annuitant can lock (i.e., allocate to fixed) or unlock (i.e., allocate to variable) any portion of the payment stream at anytime. Most immediate annuity vendors, however, impose some restrictions on mobility, and are classified somewhere between institutionalized buy-and-hold and dynamic asset allocations.

Therefore, to avoid being detained by the minutiae on the individual products, in this note we will assume that the annuitant must pick an asset mix upon retirement that is non-reversible, but that the portfolio can be rebalanced back to the same mix. In other words, we capture the salient features of the product restrictions by assuming a so-called *constant proportional* strategy.

To place this in context, one can think of a buy-and-hold restriction as implying that wealth is constrained to $W_t = \alpha X_t + (1-\alpha)Y_t$, where X_t and Y_t are the stochastic asset class values at time t. In contrast, full dynamic asset mobility implies that $\mathrm{d}W_t = \alpha(\omega, t)\,\mathrm{d}X_t + (1-\alpha(\omega, t))\,\mathrm{d}Y_t$, which is an obviously much richer set of strategies, given the time and state dependence of $\alpha(\omega, t)$. We, however, will assume a compromising constant rebalanced assumption that implies $\mathrm{d}W_t = \alpha\,\mathrm{d}X_t + (1-\alpha)\,\mathrm{d}Y_t$.

In addition, we stress that in contrast to mutual funds portfolios, taxation does not affect the allocation decision since all investment gains are treated equally as ordinary income, in the two sub-accounts.[2] We will henceforth focus on the pre-tax amounts of income.

3.1. Fixed immediate annuity

We assume that at retirement age x, the consumer has W units of wealth. If the entire W is used to purchase an FIA, the consumer will be entitled to a constant lifelong income of $c^r = W/a_x(r)$ per period, where

$$a_x(r) = \int_0^\infty \mathrm{e}^{-rt}({}_t p_x)\,\mathrm{d}t, \tag{4}$$

r is the risk-free interest rate, and ${}_t p_x$ the conditional probability that an individual of age x will survive for another t years. More generally, it is well known that

$${}_t p_x = \mathrm{e}^{-\int_x^{x+t} \lambda_s\,\mathrm{d}s}, \tag{5}$$

where λ_s is the instantaneous hazard rate. See Bowers et al. (1986) for the relationship between hazard rates, survival probabilities and mortality laws.

Therefore, $a_x(r)$ is the present value of \$ 1 per period of lifetime income. The higher the risk-free rate r, the lower is the value of $a_x(r)$ and the higher is the lifelong payment c^r.[3]

The amount of lifelong periodical income, c^r, also depends on the assumption on the distribution of the individual's remaining lifetime. We will discuss two separate cases. The first case is when the remaining lifetime is exponentially distributed. The second case is when the distribution belongs to the Gompertz–Makeham family.

3.1.1. Fixed immediate annuity under exponential mortality

If the remaining lifetime is assumed to be exponentially distributed, then ${}_t p_x = \mathrm{e}^{-\lambda t}$ and we obtain a very tractable expression:

$$a_x(r) = \int_0^\infty \mathrm{e}^{-rt}(\mathrm{e}^{-\lambda t})\,\mathrm{d}t = \frac{1}{\lambda + r}, \tag{6}$$

where $\lambda > 0$ is the constant instantaneous force of mortality. In this case, the life expectancy of an individual is $1/\lambda$, which is decreasing in λ and is independent of the current age x. The FIA's payment at time t is therefore

$$c_t^r = (\lambda + r)W, \tag{7}$$

[2] If the annuity is held within a qualified (tax-sheltered) plan, the entire payment is taxable. Otherwise, only a fraction will be subject to ordinary income taxation.

[3] This general pricing scheme is consistent with an arbitrage-free framework, as shown most recently by Carriere (1999). Note that we ignore any insurance loads or expenses charged upon annuitization. Clearly, these fees will act to increase the price of $a_x(r)$, and hence decrease c^r.

which increases in both r and λ. Intuitively, the higher the force of mortality, the lower is the individual's life expectancy, and, therefore, the higher is the periodic annuity payment.

3.1.2. Fixed immediate annuity under Gompertz–Makeham mortality

In the event of Gompertz–Makeham mortality, the force of mortality obeys the law,[4]

$$\lambda_x = \lambda + \frac{1}{b}\,\mathrm{e}^{(x-m)/b}, \tag{8}$$

where λ is a positive constant, which might capture non-age related accidental deaths, and m and b are modal and scaling parameters of the distribution, respectively. We adopt Carriere's (1994) notation for the Gompertz specification, and then augment it by the constant to create a Gompertz–Makeham law. This implies that the survival probability:

$$\begin{aligned} {}_t p_x &= \exp\left\{-\lambda t - \frac{1}{b}\int_x^{x+t} \mathrm{e}^{(s-m)/b}\,\mathrm{d}s\right\} = \exp\{-\lambda t - (\mathrm{e}^{(x+t-m)/b} - \mathrm{e}^{(x-m)/b})\} \\ &= \exp\{-\lambda t + \mathrm{e}^{(x-m)/b}(1-\mathrm{e}^{t/b})\} = \exp\{-\lambda t + b(\lambda_x - \lambda)(1-\mathrm{e}^{t/b})\}. \end{aligned} \tag{9}$$

In the pure Gompertz case, $\lambda = 0$ and the expression for the survival probability reduces to:[5]

$${}_t p_x = \exp\{b\lambda_x(1-\mathrm{e}^{t/b})\}. \tag{10}$$

Consequently, after substituting Eq. (9) into Eq. (4), the annuity factor—i.e. the price of \$ 1 for life—can be expressed as

$$\begin{aligned} {}_u a_x(r) &= \int_u^{\infty} \exp\{-(\lambda + r)t + b(\lambda_x - \lambda)(1-\mathrm{e}^{t/b})\}\,\mathrm{d}t \\ &= \exp\{b(\lambda_x - \lambda)\}\int_u^{\infty} \exp\{-(\lambda + r)t - b(\lambda_x - \lambda)\,\mathrm{e}^{t/b}\}\,\mathrm{d}t, \end{aligned} \tag{11}$$

where—with some economic abuse of actuarial notations—the left subscript in the symbol ${}_u a_x(r)$ denotes the general case where income starts at time $u \geq 0$. We now substitute the change-of-variable $s = \exp\{t/b\}$, and $\mathrm{d}s = \mathrm{d}t\exp\{t/b\}/b$, so that $\mathrm{d}s/s = \mathrm{d}t/b$, and $s^b = \exp\{t\}$, which leaves us with

$${}_u a_x(r) = b\exp\{b(\lambda_x - \lambda)\}\int_{\exp\{u/b\}}^{\infty} s^{-(\lambda + r)b-1}\exp\{-b(\lambda_x - \lambda)s\}\,\mathrm{d}s. \tag{12}$$

Finally, we substitute a second change-of-variable and let $w = b(\lambda_x - \lambda)s$, so that $\mathrm{d}w = b(\lambda_x - \lambda)\,\mathrm{d}s$, and therefore

$$\begin{aligned} {}_u a_x(r) &= \frac{b(b\lambda_x - b\lambda)^{(\lambda + r)b+1}}{b(\lambda_x - \lambda)}\exp\{b(\lambda_x - \lambda)\}\int_{b(\lambda_x - \lambda)\exp\{u/b\}}^{\infty} w^{-(\lambda + r)b-1}\exp\{-w\}\,\mathrm{d}w \\ &= b(b\lambda_x - b\lambda)^{(\lambda + r)b}\exp\{b(\lambda_x - \lambda)\}\Gamma\left(-(\lambda + r)b, b(\lambda_x - \lambda)\exp\left\{\frac{u}{b}\right\}\right) \\ &= \frac{b\Gamma(-(\lambda + r)b, b(\lambda_x - \lambda)\exp\{u/b\})}{\exp\{(m-x)(\lambda + r) + b(\lambda - \lambda_x)\}}, \end{aligned} \tag{13}$$

[4] Note that in some sources (e.g. Bowers et al., 1986), the Gompertz–Makeham law is expressed as: $A + Bc^x$, which is clearly synonymous with our formulation using $A = \lambda$, $B = \exp\{-m/b\}/b$ and $c = \exp\{1/b\}$.

[5] As an example for the pure Gompertz case, suppose that $m = 88.18$, $b = 10.5$, and the current age is $x = 65$. We find that the probability that a 65-year-old individual will survive for another 10 years, denoted by ${}_{10}p_{65}$, is 0.83942. The probability that this individual will survive for another 35 years, ${}_{35}p_{65}$, is 0.05117.

where the last equality is obtained by realizing that $(b\lambda_x - b\lambda)^{(\lambda+r)b} = \exp\{(x-m)(\lambda+r)\}$, and $\Gamma(\cdot,\cdot)$ is an incomplete gamma function defined as

$$\Gamma(x,y) = \int_y^\infty \mathrm{e}^{-t} t^{(x-1)}\,\mathrm{d}t. \tag{14}$$

Finally, when $\lambda = 0$ (which is the pure Gompertz case), and $u = 0$ (which is the immediate annuity case), we have that

$$a_x(r) = \frac{b\Gamma(-rb, b\lambda_x)}{\exp\{(m-x)r - b\lambda_x\}}. \tag{15}$$

We note that by letting $r = 0$ in Eq. (15), we obtain the individual's life expectancy:

$$a_x(0) = e_x = \exp\{b\lambda_x\} b\Gamma(0, b\lambda_x). \tag{16}$$

In any event, the constant lifelong periodical income from the FIA under the pure Gompertz distribution is

$$c_t^r = \frac{W}{a_x(r)} = \frac{\exp\{(m-x)r - b\lambda_x\}}{b\Gamma(-rb, b\lambda_x)} W. \tag{17}$$

3.2. Variable immediate annuity

The alternative to an FIA is a VIA. An investment of W dollars in a VIA will entitle the individual to a lifelong payment of $W/a_x(h)$ *units* per period, where

$$a_x(h) = \int_0^\infty \mathrm{e}^{-ht}({}_t p_x)\,\mathrm{d}t. \tag{18}$$

Note the distinction between this case and the FIA. Instead of obtaining a fixed number of *dollars* for life, the individual in this case is entitled to a fixed number of units for life. The discount rate h, which is arbitrary—but usually in the vicinity of r—is called the *assumed interest rate* (AIR) in the insurance lexicon.

Each payment unit entitles the individual to a variable (i.e., random) payment that depends on the performance of the chosen underlying asset (typically, an equity fund) vis a vis the AIR. If the return on the underlying asset in any one period is less than the AIR, the variable payment will decrease. If, on the other hand, the return on the asset is greater than the AIR, the variable payment will increase. Formally, if the underlying asset is the same risky asset whose price dynamics are as discussed in Eq. (1), then the VIA's dollar income at time t, c_t^μ, is

$$c_t^\mu = \frac{W}{a_x(h)} \exp\left\{\left(\mu - h - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}, \tag{19}$$

where B_t, μ, σ are as defined before. The income c_t^μ is random due to the Brownian motion term B_t.

3.2.1. Variable immediate annuity under exponential mortality

In the case of exponential mortality, $a_x(h) = \lambda + h$ and the income flow becomes

$$c_t^\mu = (\lambda + h) W \exp\{(\mu - h - \tfrac{1}{2}\sigma^2)t + \sigma B_t\}. \tag{20}$$

The expression for this VA income may seem obscure at first, but a comparison to the income from an FIA is quite illustrative. For example, if the AIR is equal to the risk-free rate (i.e., $h = r$), the individual is entitled to $(\lambda + r)W$ units. If the chosen underlying asset were a risk-free asset, then $\mu = r$ and $\sigma = 0$, and each unit would pay off \$ 1. Therefore, the total income would be exactly the same as in the FIA case: $c^r = (\lambda + r)W$ per period for life.

The higher the assumed interest rate h, the greater is the value of $(\lambda + h)W$. In other words, more units are acquired. However, this is not a free lunch, since the drift of the return process will be lower, and therefore the payment from each unit, $\exp\{(\mu - h - \sigma^2/2)t + \sigma B_t\}$, will be reduced. In practice, all values of h are actuarially equivalent.

3.2.2. Variable immediate annuity under Gompertz mortality

In the event of pure Gompertz mortality, the income flow becomes

$$c_t^{\mu} = \frac{\exp\{(m-x)h - b\lambda_x\}}{b\Gamma(-hb, b\lambda_x)} W \exp\left\{\left(\mu - h - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\}, \tag{21}$$

where we make use of the analogy between $a_x(h)$ in this case and $a_x(r)$ in Eq. (17).

4. Post-retirement asset allocation

The choice between a fixed and a variable payment stream is not a mutually exclusive one. The individual can select any combination of fixed and variable payment streams. This, of course, is where the asset allocation decision comes into play. If we let α denote the proportional allocation to the risky category through the purchase of the VIA, and $(1-\alpha)$ the allocation to the risk-free category through the purchase of an FIA, then the annuity payments will be a mix of returns from both annuities. Furthermore, by selecting the AIR of the VIA such that it is exactly equal to the risk-free rate, we can write a general format for the combined annuity flow:

$$c_t = \frac{W}{a_x(r)} \exp\left\{\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right)t + \alpha\sigma B_t\right\}, \tag{22}$$

which is obtained by solving:

$$\frac{\mathrm{d}c_t}{c_t} = \alpha\frac{\mathrm{d}c_t^{\mu}}{c_t^{\mu}} + (1-\alpha)\frac{\mathrm{d}c_t^{r}}{c_t^{r}}. \tag{23}$$

Note that c_t is an implicit function of α, which represents the fraction of the entire cash flow rate (in continuous-time) that is accounted for by the VIA. One can easily confirm that when $\alpha = 1$, the annuity flow is completely variable. When $\alpha = 0$, then $c_t = W/a_x(r)$, and the annuity payment is fixed. Any number between 0 and 1, will result in a degree of variability.

While the individual is in the consumption stage, he/she will maximize the discounted expected utility of consumption which we assume, again, to have a functional form of $u(x) = (1/(1-\gamma))x^{1-\gamma}$ with the discount factor of $\mathrm{e}^{-\rho s}$.

It is worth noting that the discounted utility of consumption (not the expectation) is the random variable

$$U(\alpha) = \int_0^{\tilde{T}} \mathrm{e}^{-\rho t}\frac{1}{1-\gamma}c_t^{1-\gamma}\,\mathrm{d}t, \tag{24}$$

which, in the general case, can also be written as

$$\mathbf{X} := \int_0^{\tilde{T}} a\,\mathrm{e}^{bt + cB_t}\,\mathrm{d}t. \tag{25}$$

The random variable $\mathbf{X}$, and its distribution, have been studied extensively in the economics and actuarial literature. See, for example Merton (1975), Dufresne (1990), Majumdar and Radner (1991), Yor (1992), Paulsen (1993), DeSchepper et al. (1992, 1994), Vanneste et al. (1994, 1997), Gjessing and Paulsen (1997), Milevsky (1997). However, we do not pursue this literature since we are interested in the *expected* utility as opposed to the entire random variable.

The discounted expected utility of consumption is therefore

$$EU(\alpha) = E\left[\int_0^{\tilde{T}} \mathrm{e}^{-\rho t}\frac{1}{1-\gamma}c_t^{1-\gamma}\,\mathrm{d}t\right]. \tag{26}$$

We now make use of the fact that

$$E\left[\int_0^{\tilde{T}} g(B_t)\,\mathrm{d}t\right] = E\left[\int_0^{\infty} g(B_t)1_{\{\tilde{T}>t\}}\,\mathrm{d}t\right] = E\left[\int_0^{\infty} g(B_t)({}_t p_x)\,\mathrm{d}t\right], \tag{27}$$

when $\tilde{T}$ is assumed to be independent of B_t (i.e., stock market performance does not affect the individual's health) and $1_{\{\tilde{T}>t\}}$ denotes the indicator function of the event that death occurs after t. We obtain that

$$EU(\alpha) = E\left[\int_0^{\infty} \mathrm{e}^{-\rho t}\frac{1}{1-\gamma}c_t^{1-\gamma}({}_t p_x)\,\mathrm{d}t\right]. \tag{28}$$

We now proceed with the optimal allocation under the two mortality distributions, exponential and Gompertz–Makeham.

4.1. Optimal allocation under exponential mortality

Using the value of c_t from Eq. (22) and the value of $a_x(r)$ from Eq. (6), we obtain the expected utility function in case of exponential mortality:

$$EU(\alpha) = E\left[\int_0^{\infty} \mathrm{e}^{-\lambda t}\,\mathrm{e}^{-\rho t}\frac{1}{1-\gamma}(\lambda+r)^{1-\gamma}W^{1-\gamma}\left(\exp\left\{\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right)t + \alpha\sigma B_t\right\}\right)^{1-\gamma}\mathrm{d}t\right], \tag{29}$$

which can be simplified to

$$EU(\alpha) = \frac{(\lambda+r)^{1-\gamma}W^{1-\gamma}}{1-\gamma}E\left[\int_0^{\infty}\exp\left\{-\lambda t - \rho t + (1-\gamma)\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right)t + (1-\gamma)\alpha\sigma B_t\right\}\mathrm{d}t\right]. \tag{30}$$

Using Fubini's theorem and the moment generating function of the standard normal variable, one can show that:[6]

$$E\left[\int_0^{\infty}\mathrm{e}^{-at+cB_t}\,\mathrm{d}t\right] = \int_0^{\infty}\mathrm{e}^{-at}E[\mathrm{e}^{cB_t}]\,\mathrm{d}t = \int_0^{\infty}\mathrm{e}^{-at}\,\mathrm{e}^{(1/2)c^2t}\,\mathrm{d}t = \int_0^{\infty}\mathrm{e}^{-(a-(1/2)c^2)t}\,\mathrm{d}t = \frac{1}{a-(1/2)c^2} \tag{31}$$

for any $a > c^2/2$. Using these facts, the integral in Eq. (30) can be "solved" to yield

$$EU(\alpha) = \frac{(\lambda+r)^{1-\gamma}W^{1-\gamma}}{(1-\gamma)(\lambda+\rho-(1-\gamma)\alpha(\mu-r-\alpha\sigma^2/2)-(1/2)((1-\gamma)\alpha\sigma)^2)}, \tag{32}$$

provided that

$$\lambda+\rho-(1-\gamma)\alpha(\mu-r-\tfrac{1}{2}\alpha\sigma^2) > \tfrac{1}{2}((1-\gamma)\alpha\sigma)^2. \tag{33}$$

We label Eq. (33) the *convergence criteria*. A sufficient condition for the convergence criteria is that $\mu < \lambda+\rho+r$, since $\gamma > 0$, and the allocation ranges from 0 to 1, i.e., $0 \le \alpha \le 1$.

Eq. (32) provides the expected utility of consumption as a function of the variable α which represents the choice between fixed versus variable immediate annuities. The maximum expected utility is obtained by solving the first-order condition, $EU'(\alpha^{**}) = 0$. Differentiating Eq. (32) gives

$$EU'(\alpha) = \frac{((\lambda+h)W)^{1-\gamma}(\mu-r-\alpha\sigma^2+(1-\gamma)\alpha\sigma^2)}{(\lambda+\rho-(1-\gamma)\alpha(\mu-r-\alpha\sigma^2/2)-(1/2)((1-\gamma)\alpha\sigma)^2)^2}. \tag{34}$$

[6] Eq. (31) was originally derived and then extended by Beekman and Fuelling (1990, 1991).

Setting this expression (numerator) equal to zero, and solving for α, leads to

$$\alpha^{**} = \min\left[\frac{\mu - r}{\gamma\sigma^2}, 1\right]. \tag{35}$$

The concavity of the function is confirmed, i.e. $E[U''(\alpha^{**})] < 0$, when the convergence criteria is satisfied.

Eq. (35) is identical to Eq. (3) and the optimal allocation is independent of either the subjective discount rate ρ, or the force of mortality λ. This is analogous to the time-invariant result in the pre-retirement scenario. This result shows that *even* in the presence of annuities, which provide protection against outliving one's wealth, the allocation will remain the same in the post-retirement period as in the pre-retirement period.

4.2. Optimal allocation under Gompertz mortality

Using the value of c_t from Eq. (22) and the value of $a_x(r)$ from Eq. (15), we obtain the expected utility function in case of (pure) Gompertz mortality:

$$EU(\alpha) = E\left[\int_0^\infty ({}_t p_x)\, \mathrm{e}^{-\rho t} \frac{1}{1-\gamma}\left(\frac{\exp\{(m-x)r - b\lambda_x\}}{b\Gamma(-rb, b\lambda_x)}\right)^{1-\gamma} \times W^{1-\gamma}\left(\exp\left\{\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right)t + \alpha\sigma B_t\right\}\right)^{1-\gamma} \mathrm{d}t\right], \tag{36}$$

which can be simplified to

$$EU(\alpha) = \frac{1}{1-\gamma}\left(\frac{\exp\{(m-x)r - b\lambda_x\}}{b\Gamma(-rb, b\lambda_x)}\right)^{1-\gamma} \times W^{1-\gamma} E\left[\int_0^\infty ({}_t p_x) \exp\left\{-\rho t + (1-\gamma)\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right)t + (1-\gamma)\alpha\sigma B_t\right\} \mathrm{d}t\right]. \tag{37}$$

Using Fubini's theorem, we bring the expectation inside, and recognize the moment generating function of the normal variate to obtain

$$EU(\alpha) = \frac{1}{1-\gamma}\left(\frac{\exp\{(m-x)r - b\lambda_x\}}{b\Gamma(-rb, b\lambda_x)}\right)^{1-\gamma} \times W^{1-\gamma}\left[\int_0^\infty ({}_t p_x) \exp\left\{-\rho t + (1-\gamma)\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right)t + \frac{1}{2}((1-\gamma)\alpha\sigma)^2 t\right\} \mathrm{d}t\right], \tag{38}$$

which can be simplified to

$$EU(\alpha) = \frac{1}{1-\gamma}\frac{\exp\{(1-\gamma)(m-x)r - (1-\gamma)b\lambda_x\}W^{1-\gamma}}{b^{1-\gamma}\Gamma^{1-\gamma}(-rb, b\lambda_x)} \times \left[\int_0^\infty ({}_t p_x) \exp\left\{-\left(\rho - (1-\gamma)\alpha\left(\mu - r - \frac{\alpha\sigma^2}{2}\right) - \frac{1}{2}((1-\gamma)\alpha\sigma)^2\right)t\right\} \mathrm{d}t\right]. \tag{39}$$

Thus, if we let

$$\xi = \rho - (1-\gamma)\alpha(\mu - r - \tfrac{1}{2}\alpha\sigma^2) - \tfrac{1}{2}((1-\gamma)\alpha\sigma)^2, \tag{40}$$

then the integral portion in Eq. (39) is indeed $a_x(\xi)$, or the present value of \$ 1 per period of lifetime income, evaluated at an interest rate ξ. From Eq. (15) this implies that

$$EU(\alpha) = \frac{1}{1-\gamma}\frac{\exp\{(1-\gamma)(m-x)r - (1-\gamma)b\lambda_x\}W^{1-\gamma}}{b^{1-\gamma}\Gamma^{1-\gamma}(-rb, b\lambda_x)}\frac{b\Gamma(-\xi b, b\lambda_x)}{\exp\{(m-x)\xi - b\lambda_x\}}, \tag{41}$$

which is the expected utility as an implicit function of the asset allocation parameter α.

After some tedious algebra (and the assistance of a symbolic computational language, Maple) we confirm, with no surprise, that the first derivative of Eq. (41) evaluated at $\alpha^{**} = \mu - r/\gamma\sigma^2$ is zero, whereas the second derivative at the same point, is negative. Thus, similar to the exponential case in the previous section, our asset allocation invariance result is robust to a variety of mortality distributions.

5. Caveats, generalizations and conclusions

In this note, we have developed a continuous-time stochastic model for the income derived from a fluctuating immediate annuity. We use this model to obtain the optimal allocation between risky and risk-free assets during the payout phase of a VA contract, which normally coincides with retirement. This optimal allocation is contrasted with its counterpart in the accumulation phase. We show that under CRRA utility and GBM dynamics, the optimal allocation to the risky asset remains the same upon the transition to the payout phase. We also prove that this allocation is robust to different assumptions on mortality and thus illustrate the universality of the Merton (1971) optimum, even when applied to mortality-contingent claims.

Note that we have not addressed the issues of *when* to annuitize, as well as *how much* to annuitize, but rather focused exclusively on the asset allocation *within* the annuity contract assuming it is *all* annuitized. However, there is some preliminary theoretical evidence to suggest that decreasing relative risk aversion preferences, insurance fees and expenses, strong bequest motives and pre-existing (such as government) annuities, might dramatically alter the optimal mix. We leave this agenda for further research in Milevsky and Young (2001), Charupat et al. (2001).

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