

# Self-Annuitization and Ruin in Retirement

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# Self-Annuitization and Ruin in Retirement

## ABSTRACT

At retirement, most individuals face a choice between voluntary annuitization and discretionary management of assets with systematic withdrawals for consumption purposes. Annuitization – buying a life annuity from an insurance company – assures a lifelong consumption stream that can not be outlived, but at the expense of a complete loss of liquidity. On the other hand, discretionary management and consumption from assets – self annuitization – preserves flexibility but with the distinct risk that a constant standard of living will not be maintainable.

In this paper we compute the *lifetime* and *eventual* probability of ruin (PoR) for an individual who wishes to consume a fixed periodic amount – a self constructed annuity – from an initial endowment invested in a portfolio earning a stochastic (lognormal) rate of return. The *lifetime* probability of ruin is the probability that net wealth will hit zero prior to a stochastic date of death. The *eventual* PoR is the probability that net wealth will ever hit zero for an infinitely lived individual.

We demonstrate that the probability of ruin can be represented as the probability that the *stochastic present value* (SPV) of consumption is greater than the initial investable wealth. The lifetime and eventual probabilities of ruin are then obtained by evaluating one-minus the cumulative density function of the SPV at the initial wealth level. In the eventual case we offer a precise analytical solution since the SPV is known to be Reciprocal Gamma distributed. For the lifetime case – using the Gompertz law of mortality – we provide two approximations. Both involve ‘moment matching’ techniques that are motivated by results in Arithmetic Asian option pricing theory. We verify the accuracy of these approximations using Monte Carlo simulations. Finally, a numerical case study is provided using Canadian mortality and capital market parameters. It appears that the lifetime probability of ruin – for a consumption rate that is equal to the life annuity payout – is at its lowest with a well diversified portfolio.

**Keywords:** *Asset Allocation, Retirement Planning, Value-at-Risk, Probability of Shortfall.*

# 1 Introduction

At retirement, most individuals face a choice between voluntary annuitization and discretionary management of assets with systematic withdrawals for consumption purposes. For example, in Canada this occurs at age 69 when individuals must either annuitize their tax-sheltered savings or create a systematic withdrawal plan. In the U.K. these plans are known as pension draw-downs.

Annuitization assures a lifelong consumption stream that can not be outlived, but at the expense of a complete loss of liquidity. On the other hand, discretionary management and consumption from assets – self-annuitization – preserves flexibility but with the distinct risk that a constant standard of living will not be maintainable. A retiree who decides to forgo the life annuity and instead withdraw a fixed periodic amount from wealth will find it difficult to compete with the very high ‘mortality credits’ at advanced ages.

To assist individuals with this decision, we compute the *lifetime* and *eventual* probability of ruin (PoR) for an individual who wishes replicate a synthetic life annuity, from an initial endowment invested in a portfolio earning a stochastic (lognormal) rate of return. The lifetime probability of ruin is the probability that net-wealth will hit zero prior to a stochastic date of death. The eventual PoR is the probability that net-wealth will ever hit zero for an infinitely lived investor like an endowment fund.

We demonstrate that the probability of ruin can be represented as the probability that a quantity we call the *stochastic present value* (SPV) of consumption is greater than initial wealth. The lifetime and eventual probabilities of ruin are then obtained by evaluating one-minus the cumulative density function (cdf) of the SPV at the initial wealth level. In the eventual case we offer a precise analytical solution since the SPV is known to be Reciprocal Gamma distributed. For the lifetime case we provide two approximations, using the Gompertz law of mortality. Both involve ‘moment matching’ techniques that are motivated by results in Arithmetic Asian option pricing theory. We verify the accuracy of these approximations using Monte Carlo simulations. Finally, a numerical case study is provided using Canadian mortality and capital market parameters. It appears that the lifetime probability of ruin for a consumption rate that is equal to the life annuity payout is at its lowest with a well diversified portfolio. This agrees with earlier simulation-based results of Milevsky, Ho and Robinson (1997).

The actuarial intuition is that the probability of ruin can be formulated as the probability that the SPV – basically a life annuity or perpetuity – is greater than the initial wealth available to support the consumption. Thus, in our framework, an individual is currently aged ( $x$ ) with an initial wealth of  $W_0 = w$  and a desired life-long consumption stream of  $k$  real dollars per annum. The constant  $k$  can be any arbitrary amount that the individual chooses, but presumably it is influenced by the available annuity rates in the market.

In a deterministic world, with fixed time of death  $T$  and a fixed (real) interest rate  $r$ , the present value

of the desired consumption stream is trivially calculated as:

$$k\bar{a}_{T|r} = k \int_0^T e^{-rt} dt = \frac{k(1 - e^{-rT})}{r} \quad (1)$$

If the expression in equation (1) is greater than the initial wealth  $w$ , the individual does not have enough to support the desired consumption stream and ruin occurs with probability one. Likewise, when  $T = \infty$ , equation (1) becomes  $k/r$ , which is the sum needed to fund a perpetuity of  $k$  dollars per annum.

We extend this concept to a stochastic environment where both the time of death and the rate of return on investment are random. The stochastic analogue to the deterministic present value of consumption is therefore the *stochastic present value* (SPV). This paper is also related to the recent work by Milevsky (1998) in which the author analyzes the probability of ‘beating’ the rate of return from a life annuity. While that paper focuses on the specific issue of life annuities and their returns, this paper looks at the general question of self-annuitization.

From a larger perspective, we view this research as an extension of the literature on ruin probabilities in insurance, such as the work by Pentkainen (1980), Panjer (1986), and others. The main distinction, of course, is that we focus on ‘ruin’ from a personal perspective, and they do so on the company-firm level. In particular we assume that the (consumption) ‘claims’ are deterministic and the (investment returns) ‘premiums’ are stochastic.

## 1.1 Agenda

The remainder of this paper is organized as follows. Section 2 describes the model for mortality and investment returns. Section 3 illustrates the connection between the probability of ruin, the stochastic present value and the connection to Arithmetic Asian option pricing. Section 4 provides the analytical result for eventual probability of ruin (EPoR) and the ‘moment matching’ technique for lifetime probability of ruin (LPoR). Section 5 verifies the accuracy of our approximation using Monte Carlo simulations and provides a numerical ‘case study’ using Canadian data. The main contributions of the paper are summarized in Section 6, where we discuss the relation to corporate risk management and Value-at-Risk. The notation used in the paper is summarized prior to the appendices.

## 2 Model for Investment and Mortality Risk

### 2.1 Investments

We assume the standard geometric Brownian motion (GBM) model of market prices - first implemented by Robert Merton<sup>1</sup> in the late 1960’s - in which the real (after-inflation) value of the investment portfolio

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<sup>1</sup>See Merton (1993) for details and a collection of his classic articles.

obeys the stochastic differential equation (SDE) defined by:

$$dV_t = \mu V_t dt + \sigma V_t dB_t, \quad V_0 = 1, \quad (2)$$

where  $B_t$  is a standard one-dimensional Brownian motion,  $\mu$  is the rate of return (also known as drift co-efficient) and  $\sigma$  is the volatility (also known as the diffusion coefficient). The only parameter restriction imposed is that  $\mu - \sigma^2/2 > 0$ , which is known as a growth condition. In our model, the retiree invests his initial wealth, denoted by  $w$ , in a portfolio satisfying equation (2) and then consumes a fixed real (after-inflation) amount  $k$ , per unit of time. This is what we mean by self-annuitization; namely, the retiree creates a constant consumption stream similar to a life annuity. A life annuity would yield a consumption stream of  $w/\bar{a}_x$ , while the quantity  $k$  may be greater or less than  $w/\bar{a}_x$ . By construction, the real net-wealth process of the retiree will obey the SDE:

$$dW_t = (\mu W_t - k) dt + \sigma W_t dB_t, \quad W_0 = w, \quad (3)$$

where  $\mu, \sigma, B_t$  are as in equation (2). The net-wealth process defined by equation (3) has a drift  $(\mu W_t - k)$ , that *may* become negative if  $k$  is large enough relative to  $\mu W_t$ . This, in turn, implies that the process  $W_t$ , *may* eventually hit zero, in stark contrast to the classical geometric Brownian motion which is bounded away from zero in finite time. Our intention, now, is to (a) compute the probability that  $W_t$  will ever hit zero and then (b) compute the probability that  $W_t$  will hit zero while the retiree is still alive. The former will be labeled the ‘eventual’ or ‘endowment’ probability of ruin (EPoR), and the latter will be called the ‘lifetime’ probability of ruin (LPoR). EPoR is an upper bound for LPoR, and the younger the individual, the closer the two numbers.

**Theorem #1:** The net-wealth process  $W_t$ , defined by equation (3), can be solved explicitly to yield:

$$W_t = V_t \left[ w - k \int_0^t (V_s)^{-1} ds \right], \quad (4)$$

where the  $V_s$  is equal to:

$$V_s = \exp\left\{(\mu - \frac{1}{2}\sigma^2)s + \sigma B_s\right\}. \quad (5)$$

The *fundamental process*  $V_s$ , can be identified as the (GBM) solution to equation (2).

**Proof:** See the book by Karatzas and Shreve (1992, page 361). The proof requires a basic application of the method of variation of coefficients. The solution can be confirmed by applying Ito’s Lemma to equation (4) and thus recovering equation (3).

## 2.2 Mortality

Following the actuarial literature on parametric mortality (see for example Carrière (1992)), we assume a Gompertz law for mortality. In this model, the probability of survival to age  $(x + t)$ , conditional on a life

at age  $(x)$ , is denoted by  ${}_t p_x$  and defined as:

$${}_t p_x = \mathbf{P}(\tilde{T} \geq t \mid m, l, x) = \exp \left\{ \exp \left\{ \frac{x-m}{l} \right\} \left( 1 - \exp \left\{ \frac{t}{l} \right\} \right) \right\}, \quad (6)$$

where  $m$  is the mode,  $l$  is the scale parameter and  $\tilde{T}$  denotes the remaining time-until-death random variable. For example, when the mode of life is  $m = 80$  and the ‘‘scale’’ of life is  $l = 10$ , equation (6) stipulates that the probability a 65 year old, lives to age 85, is:  $\mathbf{P}(\tilde{T} \geq 20 \mid 80, 10, 65) = 0.2404$ . The probability that a 75 year old lives to age 85, is:  $\mathbf{P}(\tilde{T} \geq 10 \mid 80, 10, 75) = 0.3527$ . Naturally, the probability of reaching age 85 increase the older you are. Letting  $t \rightarrow \infty$  in equation (6), with a finite value for  $m$  and  $l$ , results in  $({}_t p_x) \rightarrow 0$  which confirms the natural boundary condition of human life. Likewise, for all values of  $t$ , when  $m \rightarrow \infty$  in equation (6), we get that  $({}_t p_x) \rightarrow 1$ . Thus, the Gompertz distribution nests the endowment (eventual) case and the notation  ${}_t p_x$  can be used for LPoR and EPoR, without any loss of generality.

In fact, according to Carrière (1992) and Frees, Carrière and Valdez (1996), a properly calibrated Gompertz mortality function serves as an excellent approximation for mortality, especially at retirement ages. In Section 5, where we provide some numerical values for LPoR and EPoR, we will use a Gompertz distribution that is ‘fitted’ to a discrete mortality table at various initial ages using a non-linear optimization procedure.

### 3 The Probability of Ruin and Stochastic Present Values

We will now compute the probability that the net-wealth process, defined by equation (3), hits zero (i) while the retiree is still living and (ii) ever. Using formal notation, we are interested in:

$$\text{LPoR} := \mathbf{P} \left[ \inf_{0 \leq t \leq \tilde{T}} W_t \leq 0 \mid W_0 = w \right], \quad \text{EPoR} = \mathbf{P} \left[ \inf_{0 \leq t \leq \infty} W_t \leq 0 \mid W_0 = w \right]. \quad (7)$$

Before we proceed to obtain an analytic expression for EPoR and an approximation for LPoR, we state and prove the following.

**Theorem #2.** The stochastic process  $W_t$ , defined by equation (3), obeys the following property:

$$\mathbf{P} \left[ \inf_{0 \leq t \leq T} W_t \leq 0 \mid W_0 = w \right] = \mathbf{P} [W_T \leq 0 \mid W_0 = w], \quad \text{for all } T > 0 \quad (8)$$

Stated differently, the process  $W_t$  will not cross zero more than once. Once it enters the negative region, it stays there.

**Proof.** See Appendix A.

Relying on Theorem #2, which applies to any  $T$  – stochastic, finite or infinite – we can re-write the probability of ruin in equation (7) as:

$$\text{LPoR} = \mathbf{P} \left[ W_{\tilde{T}} \leq 0 \mid W_0 = w \right] = \mathbf{P} \left[ \frac{w}{k} \leq \int_0^{\tilde{T}} (V_s)^{-1} ds \right]. \quad (9)$$

and

$$\text{EPoR} = \mathbf{P}[W_\infty \leq 0 \mid W_0 = w] = \mathbf{P}\left[\frac{w}{k} \leq \int_0^\infty (V_s)^{-1} ds\right]. \quad (10)$$

Equations (9) and (10) constitute an important intermediary step in our calculations. The PoR can be expressed as the probability that the stochastic present value of (lifetime, infinite) consumption of one real dollar per annum is greater than the initial wealth to consumption ratio  $w/k$ . The connection between probability of ruin and stochastic present values has recently been analyzed by Paulsen (1993), Gjessing and Paulsen (1997) and Norberg (1995), but can be traced back to work by Harrison (1977).

Theorem #2 transforms our main research objective to obtaining a suitable density function for the stochastic variable:  $\int_0^t (V_s)^{-1} ds$ , where  $t$  may be finite, infinite or an (independent) random stopping time in  $[0, \infty)$ . In fact, the expression  $\int_0^t (V_s)^{-1} ds$  has an additional interpretation as being the present value of an annuity, which has been studied extensively in the actuarial literature.<sup>2</sup>

### 3.1 Arithmetic Asian Option Connection.

There is an interesting and useful connection between the self-annuitization probability of ruin - the SPV function - and a financial derivative security known as an Arithmetic Asian (AA) option. The AA option is a path-dependent contingent claim whose payoff at maturity is based on the *average price* of the underlying security observed over the life of the option. The payoff from a generic call option is:  $\max[S_T - K, 0]$ , where  $K$  is the exercise price and  $S_T$  is the price of the underlying security at maturity. The payoff from an Arithmetic Asian (fixed strike) call option is  $\max\left[\frac{1}{n} \sum_{i=1}^n S_i - K, 0\right]$  where  $n$  is the number of measurement-observation periods and  $S_i$  is the price of the underlying security on those discrete measurement-observation dates. When the number of measurement-observation periods is very large compared to the life-time of the option, we can extend the definition of the Arithmetic Asian option using the integral instead of the summation sign. Thus, the payoff from a continuous Arithmetic Asian option is  $\max\left[\frac{1}{T} \int_0^T S_t dt - K, 0\right]$ .

The relevance of the Arithmetic Asian option to the problem at hand is that the real-world probability that an Asian option will expire in-the-money can be formally stated as:  $\mathbf{P}\left[\int_0^T S_t dt > TK\right]$ , where  $S_t$  is the price process under the physical (statistical) measure. Therefore, if we define a new pseudo-price process  $S_t := (V_t)^{-1}$ , where  $V_t$  comes from equation (4), we get the helpful result that the real-world probability of ruin can be represented as the probability that a suitably structured AA option will expire in-the-money under the physical - as opposed<sup>3</sup> to risk-neutral - measure.

We borrow some of the techniques that have been developed to approximate  $\int_0^T S_t dt$  in the AA option pricing literature, in order to compute the lifetime probability of ruin. In the option pricing context we

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<sup>2</sup>See the research initiated by Boyle (1976), continued by Panjer and Bellhouse (1980,1981) and extended by Beekman and Fuelling (1990,1991,1993). See also the recent stream of work by Vanneste, Goovaerts, DeSchepper and Dhaene (1997), and the references therein.

<sup>3</sup>When pricing the Arithmetic Asian option, the risk-neutral measure is the only relevant probability. The No Arbitrage price of the AA option is the discounted expected value of its payoff in a risk neutral world.

are interested in the risk-neutral lower-partial moment minus a tail probability, but for this paper we simply want the real-world tail probability. Coincidentally, the structure, although not parameters, of the underlying random variable is the same.

## 4 Computing The Probability of Ruin

### 4.1 Analytic Computation of EPoR

The computation of EPoR reduces to the evaluation of (one minus) the cumulative density function (CDF) of  $\int_0^\infty (V_s)^{-1} ds$ , at  $w/c$ , where  $V_s = \exp\{(\mu - \frac{1}{2}\sigma^2)s + \sigma B_s\}$ .

**Theorem #3.** The stochastic variable:

$$X := \int_0^\infty (V_s)^{-1} ds = \int_0^\infty \exp\{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s\} ds, \quad (11)$$

obeys a Reciprocal Gamma (RG) distribution with density function:

$$\mathbf{P}(X \in dx) = \exp\left\{-\frac{1}{x\beta}\right\} \frac{x^{-(\alpha+1)}}{\Gamma(\alpha)\beta^\alpha}, \quad (12)$$

where  $\alpha = \frac{2\mu}{\sigma^2} - 1$  and  $\beta = \frac{\sigma^2}{2}$ .

**Proof:** This result can be derived in many different ways. We refer the interested reader to the papers by Dufresne (1990), Majumdar and Radner (1991), Milevsky (1998) as well as Merton (1993, page 592) for various proofs of this elegant fact.

Therefore, if we let  $\mathbf{G}(x | \alpha, \beta)$  and  $\mathbf{G}_R(x | \alpha, \beta)$  denote the CDF of the Gamma and Reciprocal Gamma respectively, we have the identity:

$$\mathbf{G}_R(x | \alpha, \beta) = \mathbf{P}(X \leq x) = \mathbf{P}(1/X > 1/x) = 1 - \mathbf{G}(1/x | \alpha, \beta). \quad (13)$$

Therefore, the numerical value of EPoR – which is  $\mathbf{G}_R(w/k | \frac{2\mu}{\sigma^2} - 1, \sigma^2/2)$  – is easily available in any commercial spreadsheet package by computing one minus the CDF of the Gamma variate.

### 4.2 Approximation for LPoR.

Unfortunately, there is no closed-form density function for  $\int_0^T (V_s)^{-1} ds$  - which is the present value in finite time. Therefore, in this section we will use ‘moment matching’ techniques suggested by Parker (1993,1994) in the insurance context and Milevsky and Posner (1998) in the Arithmetic Asian option literature. We provide two candidate approximations for  $\int_0^T (V_s)^{-1} ds$ . The first is the Reciprocal Gamma distribution and the second is called the Type II Johnson distribution, which is a member of the Johnson (1949) family. Our inspiration for the approximation techniques is that the RG distribution and the Johnson distribution have been applied<sup>4</sup> to the pricing of Arithmetic Asian options – at its essence the same problem – with

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<sup>4</sup>The interested reader is referred to Posner and Milevsky (1998) for details.

excellent results. The RG has two degrees of freedom and hence we match the first two moments. The Type II Johnson has four degrees of freedom, which leads to four moment matching. Not surprisingly, the Johnson approximation provides results that are more accurate than the RG approximation, given the two extra degrees of freedom. In fact, the Johnson approximation is remarkably indistinguishable from the Monte Carlo simulation. Of course, matching four moments requires more effort, which is why we suggest the two methods of differing complexity.

We re-iterate that with Arithmetic Asian options the ‘problem’ is computing  $E^* \left[ \left( \frac{1}{T} \int_0^T (V_s)^{-1} ds - K \right)^+ \right]$ , where  $E^*[\cdot]$  denotes the risk-neutral expectation and  $T$  is a fixed time horizon. In the context of ruin, we are interested in  $\mathbf{P} \left[ \int_0^{\tilde{T}} (V_s)^{-1} ds > w/k \right]$ , where  $\mathbf{P}[\cdot]$  is the real world measure and  $\tilde{T}$  is a random variable.

### 4.3 Moment Matching.

We now proceed to obtain the first two moments of  $\int_0^{\tilde{T}} (V_s)^{-1} ds$  and then ‘moment match’ to the RG distribution. We then expand the procedure for the third and fourth moments and match to the Johnson distribution. Our assumption remains that  $\tilde{T}$  is Gompertz distributed and independent of the Brownian motion driving  $V_s$ . To simplify notation somewhat, we denote the stochastic present value for a fixed maturity by

$$I_t = \int_0^t (V_s)^{-1} ds, \quad (14)$$

where, without loss of generality,  $I_\infty$  is the stochastic present value of perpetual consumption and  $I_{\tilde{T}}$  is the stochastic present value of lifetime consumption. Recalling the definition of  $V_s = \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) s + \sigma B_s \right\}$ , from equation (5) and using the rules for conditional expectations, we know that:

$$E \left[ I_{\tilde{T}} \right] = E \left[ E \left[ I_{\tilde{T}} \mid \tilde{T} = t \right] \right] = E \left[ E \left[ I_{\tilde{T}} \mid \mathcal{F}_\infty^B \right] \right], \quad (15)$$

where  $\mathcal{F}_\infty^B$  is the sigma field generated by the entire path of the Brownian motion. Using the moment generating function for the normal random variable  $E \left[ \exp \{ -\sigma B_s \} \right] = \exp \left\{ \frac{1}{2} \sigma^2 s \right\}$ , we get that<sup>5</sup>:

$$\begin{aligned} E \left[ I_{\tilde{T}} \right] &= E \left[ E \left[ \int_0^t \left( \exp \left\{ \left( \mu - \frac{1}{2} \sigma^2 \right) s + \sigma B_s \right\} \right)^{-1} ds \mid \tilde{T} = t \right] \right] \\ &= E \left[ \int_0^t E \left[ \exp \left\{ -\left( \mu - \frac{1}{2} \sigma^2 \right) s - \sigma B_s \right\} \right] ds \mid \tilde{T} = t \right] \\ &= E \left[ \int_0^t \exp \left\{ -\left( \mu - \frac{1}{2} \sigma^2 \right) s \right\} E \left[ \exp \{ -\sigma B_s \} \right] ds \mid \tilde{T} = t \right] \\ &= E \left[ \int_0^t \exp \left\{ -\left( \mu - \frac{1}{2} \sigma^2 \right) s \right\} \exp \left\{ \frac{1}{2} \sigma^2 s \right\} ds \mid \tilde{T} = t \right] \\ &= \int_0^\infty \exp \left\{ -\left( \mu - \sigma^2 \right) s \right\} s p_x ds. \end{aligned} \quad (16)$$

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<sup>5</sup>These results are confirmed by Boyle (1976) in discrete time and by Beekman and Fuelling (1991,1992,1993), who derived the first two moments of the annuity present value in continuous time using function space integral techniques under a variety of interest rate dynamics.

We define the function,

$$A(\xi | m, l, x) := \int_0^\infty \exp\{-\xi s\} {}_s p_x ds, \quad (17)$$

which is a form of present value operator. Indeed, after substituting the Gompertz ( ${}_s p_x$ ) from equation (6) and changing variables, we get:

$$A(\xi | m, l, x) = l \exp\left\{\exp\left\{\frac{x-m}{l}\right\} + (x-m)\xi\right\} \Gamma\left(-l\xi, \exp\left\{\frac{x-m}{l}\right\}\right), \quad (18)$$

where  $\Gamma(u, v) = \int_v^\infty e^{-t} t^{(u-1)} dt$  denotes the incomplete Gamma function. Although analytic mortality laws are currently not in vogue in the actuarial community, we prefer to use a Gompertz formulation due to its analytic tractability. Of course, the concepts underlying this paper can be applied using any mortality table.

By construction, the term  $A(\xi | m, l, x)$  in equation (17), coincides with the Gompertz price of a life-annuity under a continuously compounded force of interest  $\xi$ . Without loss of generality, since  $({}_t p_x) \rightarrow 1$  as  $m \rightarrow \infty$ , we use,

$$\lim_{m \rightarrow \infty} A(\xi | m, l, x) = \frac{1}{\xi}, \quad (19)$$

which makes equation (18) applicable to the perpetuity case as well. Going back to equation (16), the expectation of the stochastic present value of lifetime consumption is:

$$M_1 = E\left[I_{\tilde{T}}\right] = A(\mu - \sigma^2 | m, l, x), \quad (20)$$

which is the first (non-central) moment. The same technique as detailed in equation (16) can be employed to obtain all higher non-central moments of the stochastic variable  $I_{\tilde{T}}$ . Clearly, the multiple integrals get more cumbersome as the moments get higher, but the algorithm is essentially the same. For example, the second moment is:

$$M_2 = E\left[I_{\tilde{T}}^2\right] = \left(\mu/2 - \sigma^2\right) \left(A(\mu - \sigma^2 | m, l, x) - A(2\mu - 3\sigma^2 | m, l, x)\right). \quad (21)$$

The RG approximation method requires the first two moments. The expressions for the third and fourth moments of  $I_{\tilde{T}}$  - which are denoted by  $M_3$  and  $M_4$  and are quite messy<sup>6</sup> - will be used for the Johnson approximation. For the sake of completeness, we let  $m \rightarrow \infty$ , and display the first and second moments for the stochastic present value of perpetual consumption, as:

$$E[I_\infty] = \frac{1}{\mu - \sigma^2}, \quad E[I_\infty^2] = \frac{2}{(\mu - \sigma^2)(2\mu - 3\sigma^2)} \quad (22)$$

### 4.3.1 The Reciprocal Gamma Approximation.

The first and second moments of the Reciprocal Gamma random variable, as introduced in Theorem #3, are:  $1/(\beta(\alpha - 1))$  and  $1/(\beta^2(\alpha - 1)(\alpha - 2))$ , respectively. Substituting  $\alpha = \frac{2\mu}{\sigma^2} - 1$  and  $\beta = \frac{\sigma^2}{2}$ , recovers

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<sup>6</sup>Available from the authors upon request.

equation (22) for the perpetual case. In finite time however, we invert the first two moments and express the ‘fitted’ variables  $\hat{\alpha}$  and  $\hat{\beta}$  in terms of  $M_1$  and  $M_2$ . They are:

$$\hat{\alpha} = \frac{2M_2 - M_1^2}{M_2 - M_1^2}, \quad \hat{\beta} = \frac{M_2 - M_1^2}{M_2 M_1}, \quad (23)$$

where  $M_1$  and  $M_2$  are taken from equation (20) and (21) respectively. We like to make clear the distinction between  $\alpha, \beta$  and  $\hat{\alpha}, \hat{\beta}$ . The former represent the parameters to be used when computing the EPoR and are defined by theorem #3. The EPoR is exact since the distribution converges to the Reciprocal Gamma in the limit. The latter parameters  $\hat{\alpha}, \hat{\beta}$ , are used to obtain a ‘moment matched’ approximation for LPoR as per the following:

**Remark #1.** The stochastic variable:

$$X := \int_0^t (V_s)^{-1} ds = \int_0^t \exp\left\{-\left(\mu - \frac{1}{2}\sigma^2\right)s - \sigma B_s\right\} ds, \quad (24)$$

can be approximated by the Reciprocal Gamma density function and therefore:

$$\text{LPoR} \cong \mathbf{G}(k/w \mid \hat{\alpha}, \hat{\beta}) \quad (25)$$

where  $\hat{\alpha}$  and  $\hat{\beta}$  are defined by equation (23).

See the appendix of Milevsky and Posner (1998) for a proof that the RG distribution is the ‘best’ two parameter approximation for the random variable in equation (24) in the context of Arithmetic Asian options. Monte Carlo simulations in the next section will verify the ‘reasonableness’ of the approximation for our purposes

### 4.3.2 The Johnson Distribution:

If we let  $\mathbf{Z}$  denote a standard normal random variable, the Type II Johnson (1949) distribution is defined as:

$$\mathbf{X} = c + d \sinh\left(\frac{\mathbf{Z} - a}{b}\right) \iff \mathbf{Z} = a + b \sinh^{-1}\left(\frac{\mathbf{X} - c}{d}\right), \quad (26)$$

where  $a, b, c, d$  are four free parameters. Compared to the RG distribution, the  $\sinh^{-1}(\cdot)$  function may appear arbitrary in our setting. The Johnson distribution is connected to the lognormal distribution, which is the integrand in equation (5), by virtue of the implicit exponentiation of the standard normal variable, via  $\sinh(x) = (\exp\{x\} - \exp\{-x\})/2$ . Indeed, by inversion, one can show that  $\sinh^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$ .

The cdf of the Johnson distribution is:

$$\mathbf{J}(x \mid a, b, c, d) = \int_{-\infty}^{a + b \sinh^{-1}\left(\frac{x-c}{d}\right)} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{u^2}{2}\right\} du \quad (27)$$

This allows us to obtain the four moments of the Johnson distribution using the moment generating function of the normal variate. Given the four moments of the stochastic present value  $\int_0^{\tilde{T}} (V_s)^{-1} ds$ , we can then invert the four equations and ‘solve’ for the parameters  $a, b, c, d$ . The inversion process must be

done numerically - which clearly adds to the complexity - we refer the interested readers to Appendix B for more details.

**Remark #2.** The stochastic variable:

$$X := \int_0^t (V_s)^{-1} ds = \int_0^t \exp\{-(\mu - \frac{1}{2}\sigma^2)s - \sigma B_s\} ds, \quad (28)$$

can be better approximated by the Johnson distribution and therefore:

$$\text{LPoR} \cong 1 - \mathbf{J}(w/k \mid a, b, c, d) \quad (29)$$

where  $a, b, c, d$  are defined in terms of  $M_1, M_2, M_3, M_4$ .

## 5 Numerical Tests and Examples

### 5.1 Measuring the Accuracy of the Moment Approximations.

The accuracy of our moment matching technique was tested using Monte Carlo simulations. Specifically, upon the suggestion of a reviewer, we simulated the value of  $\mathbf{P}(X \geq K)$  for a variety of  $K$ s for fixed time horizons. We are interested in the effect of time and volatility on the accuracy of the approximation. The Monte Carlo simulation was constructed by discretizing time in increments of 0.01 years. Table #1 provides a few of the results.

**Tables 1a,1b,1c and 1d, go here**

We display the cases  $\sigma = 15\%$  and  $\sigma = 30\%$ , for  $T = 10$  and  $T = 25$ . A few facts are evident. First, both the two-moment RG and the four moment Johnson techniques provide ruin probabilities that are very close to the Monte Carlo (MC) values. The MC values are slightly closer to the Johnson numbers, which is to be expected given the higher moments that are targeted. It appears the RG approximations is relatively ‘worse’ for higher levels of volatility and longer time horizon. This may be surprising - at first - given the convergence to the RG distribution in the limit. However, the very large gap between LPoR and EPoR should indicate that convergence is slow.

In summary, it appears that both approximations provide reasonably accurate estimates for the lifetime probability of ruin when compared with Monte Carlo simulations.

### 5.2 Case Example of Ruin Probabilities

In this section we provide a numerical case study of the ruin probabilities - both lifetime and eventual - using Canadian mortality and historical capital market (investment return) estimates. For the lifetime case, where an approximation is required, we employ the Reciprocal Gamma distribution due to its tremendous ease of use and pedagogical appeal. Clearly, as the simulations in the previous section illustrate, the RG

approximation is slightly inferior to the Johnson method. However, the error introduced is worth the price of simplicity, given the many other assumptions made in this model.

As per the investment dynamics for the stochastic process  $V_t$ , we choose the 2+1 asset situation. In other words, we investigate the probability of ruin for a variety of investment combinations involving (i) a risk free asset, (ii) a diversified equity mutual fund and (iii) a diversified bond fund. Technically speaking we will compute EPoR and LPoR for an individual aged ( $x$ ) exhibiting Gompertz mortality ( $m, l$ ), withdrawing at a (real) rate of  $k$  per annum, facing capital market uncertainty with (real) parameters  $(\mu, \sigma)$ . The variety of parameters  $(\mu, \sigma)$  will be selected from the parameter ‘space’ of  $\mu = (1 - \gamma_e - \gamma_b)r + \gamma_e\mu_e + \gamma_b\mu_b$  and  $\sigma = \sqrt{\gamma_e^2\sigma_e^2 + \gamma_b^2\sigma_b^2}$ , where the  $e$  and  $b$  subscripts denote the parameters for the equity and bond funds respectively.

### 5.3 Canadian Mortality Data

We fit a Gompertz distribution using a non-linear optimization routine in S-plus to the *Life Tables, Canada and Provinces 1990-1992* (Statistics Canada), and obtain the parameter estimates of:  $m = 81.95, l = 10.6$  for males and  $m = 87.8, l = 9.5$  for females. For example, the probability that a 65 year old male lives to age 85, using equation (6), is:  $\mathbf{P}(\tilde{T} \geq 20 \mid 81.95, 10.6, 65) = 0.3226$ , likewise, the probability that a 65 year old female lives to age 85, using equation (6), is:  $\mathbf{P}(\tilde{T} \geq 20 \mid 87.8, 9.5, 65) = 0.5199$ . In practice, one could use any number of mortality tables - such as the IAM 1983 with some dynamic projections - depending on the context. Our purpose is simply to illustrate the technique.

### 5.4 Canadian Capital Markets Data

Ibbotson Associates, a consulting firm located in Chicago, IL, provides forecast data for long-term Canadian investment returns and volatilities. All the rates of return are continuously compounded. In this example, we focus on three asset classes – the classic cash, bonds and equities division. We assume the availability of a deterministic *real* risk-free rate of  $r = 2\%$ ; a government bond index fund with *real* parameters  $\mu_b = 3.5\%$  and  $\sigma_b = 11\%$ ; and an equity index fund with *real* parameters  $\mu_e = 8\%$  and  $\sigma_e = 19\%$ . Our numerical example assumes that the correlation coefficient between the *real* rate of return on the bond fund and the *real* rate of return on the equity fund is zero. This is consistent with empirical estimates as well.

### 5.5 Output for EPoR and LPoR

Table 2a demonstrates a single application of EPoR and LPoR for a female and a male, each aged 65, with a *real* wealth to consumption ratio,  $w/k = 14$ . This corresponds to a constant (and arbitrary) self-annuitization rate of  $\frac{1}{14}$  of initial wealth per annum. The absolute values of initial wealth and desired real consumption are irrelevant; only the ratio matters. The investment allocations are varied by increments

of 20%. Across the top of the table we show the proportion of the funds in bonds (Bonds, or  $\gamma_b$ ). Down the vertical axis we show the proportion in the equity fund (Equity, or  $\gamma_e$ ). Thus  $1 - \gamma_e - \gamma_b$  is allocated to the risk-free rate. The number in the square brackets is the eventual probability of ruin (EPoR), and the number next to it is the conditional probability of ruin (LPoR). As intuition dictates, EPoR is always much greater than LPoR. However, we do remind the reader that while the former is precise, the latter is only an approximation since we are using the Reciprocal Gamma technique.

**Table #2a,b goes here**

For example in Table 2a (the female case), a 60% allocation to bonds with a 20% allocation to equity and (the remainder) 20% allocation to cash, will result in a  $\mu = 0.041$  and  $\sigma = 0.07615$ . This, in turn, will result in a value of  $E[I_{\hat{T}}] = M_1 = 13.596$  and  $\sqrt{M_1^2 - M_2} = 5.5308$ , as per equation (20) and (21). In other words, the expected value of the discounted stochastic lifetime consumption of one dollar - the PV of the annuity - is equal to 13.596, which is slightly lower than the initial wealth-to-consumption ratio of 14. The standard deviation of the present value is 5.5308. We see that there is a strong possibility of lifetime (and obviously eventual) ruin. Finally, we compute the parameters  $\hat{\alpha} = 8.0428$  and  $\hat{\beta} = 0.010443$  using equation (23). These numbers are plugged into the cumulative density function of the Gamma distribution evaluated at the reciprocal of the wealth-to-consumption ratio, to arrive at  $G(\frac{1}{14} | 8.0428, 0.010443) = 0.371$ , which is a conditional probability of ruin of 37.12%. Thus we conclude that the  $w/k = 14$  *standard of living is sustainable*, under the above mentioned allocation, with 62.9% probability. The lowest conditional probability of ruin occurs (very roughly) with an allocation of 80% Equity, 20% Bonds and 0% Cash, and is equal to 26.7%

Table 2b displays the results for a 65 year old *male* with  $w/k = 14$ . As one would expect intuitively, the conditional probabilities of shortfall are uniformly lower for all asset allocations as a direct result of the shorter life span. The unconditional probabilities remain the same since we have not modified the capital market parameters.

Interestingly, the lowest probability of ruin occurs with a 60% allocation to equity and 40% in bonds. Other values can easily be generated using a spreadsheet and the effect of (static) asset mix on ruin probability can be investigated.

## 6 Conclusion and Directions for Future Research

Financial institutions have recently embraced risk management and the concept of Value-at-Risk (VaR). The basic idea behind VaR is to compute the maximum amount that can be lost - over a specified period of time - with a certain probability. Alternatively, VaR can be represented as the probability that the institution will suffer a loss of a certain magnitude - assuming the composition and structure of the asset and liability portfolio remains intact.

Motivated and inspired by its application in corporate institutional finance, this paper is about probabilistic risk management for the individual towards the end of the human life cycle. Specifically, we model and then provide techniques for computing the probability of ruin when an individual self-annuitizes. Self-annuitization is the process of consuming a fixed (real or nominal) amount from wealth in the hopes of replicating a life-annuity. The probability of ruin that we develop should help individuals obtain a ‘risk metric’ of their investment and consumption portfolio. Clearly, the structure and composition of this portfolio may change over time. But the ‘static’ measure of risk - similar to VaR - will demonstrate the implications of maintaining the *status quo*. We believe our contribution to the literature lies in the formulation of the ‘retirement problem’ as well as the various approximations proposed.

On a technical level, we have derived the *eventual* (EPoR) and *lifetime* (LPoR) probability of ruin, assuming lognormal asset returns and Gompertz mortality. The former is obtained analytically, while the later is approximated. The probability of ruin can be represented in terms of the *stochastic present value* of consumption - which is Reciprocal Gamma distributed for the EPoR. We approximated the LPoR via the same Reciprocal Gamma and a Johnson distribution, using ‘moment matching’ techniques.

Finally, we gave a numerical case study that illustrated how a well diversified portfolio provides the lowest *lifetime* probability of ruin.

**Table 1a:** $T = 10, \mu = 0.10, \sigma = 0.15$ 

Monte Carlo Simulation vs. Approximation Techniques

	$\mathbf{P}(X \geq 5)$	$\mathbf{P}(X \geq 6.959)$	$\mathbf{P}(X \geq 8)$	$\mathbf{P}(X \geq 11)$
Monte Carlo (Standard Error)	0.8942(0.001)	0.4397(0.002)	0.2493(0.001)	0.0269(0.002)
Johnson - Type II	0.8937	0.4402	0.2388	0.0273
Reciprocal Gamma	0.8978	0.4351	0.2352	0.0282

**Table 1b:** $T = 10, \mu = 0.10, \sigma = 0.30$ 

Monte Carlo Simulation vs. Approximation Techniques

	$\mathbf{P}(X \geq 7)$	$\mathbf{P}(X \geq 9.5163)$	$\mathbf{P}(X \geq 12)$	$\mathbf{P}(X \geq 15)$
Monte Carlo (Standard Error)	0.6179(0.002)	0.3722(0.004)	0.2221(0.003)	0.1231(0.001)
Johnson - Type II	0.6182	0.3739	0.2243	0.1243
Reciprocal Gamma	0.6187	0.3663	0.2156	0.1176

**Table 1c:** $T = 25, \mu = 0.10, \sigma = 0.15$ 

Monte Carlo Simulation vs. Approximation Techniques

	$\mathbf{P}(X \geq 9)$	$\mathbf{P}(X \geq 11.0456)$	$\mathbf{P}(X \geq 15)$	$\mathbf{P}(X \geq 18)$
Monte Carlo (Standard Error)	0.6626(0.001)	0.4119(0.002)	0.1355(0.003)	0.0556(0.004)
Johnson - Type II	0.6627	0.4120	0.1352	0.0557
Reciprocal Gamma	0.6651	0.4128	0.1335	0.0547

**Table 1d:** $T = 25, \mu = 0.10, \sigma = 0.30$ 

Monte Carlo Simulation vs. Approximation Techniques

	$P(X \geq 15)$	$P(X \geq 22.12)$	$P(X \geq 25)$	$P(X \geq 35)$
Monte Carlo (Standard Error)	0.5892(0.001)	0.3276(0.005)	0.2633(0.002)	0.1382(0.003)
Johnson - Type II	0.6063	0.3304	0.2639	0.1376
Reciprocal Gamma	0.5391	0.3126	0.2549	0.1361

**Table 2**

2A: Female age 65,  $w/c = 14$

Probability of Ruin: [EPoR], LPoR

Reciprocal Gamma Approximation

Equity vs. Bonds vs. T.Bills

<i>Equity</i> \Bonds	0%	20%	40%	60%	80%	100%
0%	[1.00].548	[1.00].518	[1.00].495	[1.00].479	[.999].472	[.993].470
20%	[.999].426	[1.00].399	[.999].380	[.996].371	[.979].370	X
40%	[.991].342	[.981].319	[.957].306	[.921].300	X	X
60%	[.884].299	[.849].281	[.811].269	X	X	X
80%	[.755].284	[.719].267	X	X	X	X
100%	[.673].285	X	X	X	X	X

2B: Male age 65,  $w/c = 14$

Probability of Ruin: [EPoR], LPoR

Reciprocal Gamma Approximation

Equity vs. Bonds vs. T.Bills

<i>Equity</i> \Bonds	0%	20%	40%	60%	80%	100%
0%	[1.00].325	[1.00].307	[1.00].295	[1.00].291	[.999].292	[.993].230
20%	[.999].250	[1.00].234	[.999].225	[.996].223	[.979].228	X
40%	[.991].206	[.981].193	[.957].186	[.921].185	X	X
60%	[.884].188	[.849].177	[.811].170	X	X	X
80%	[.755].186	[.719].176	X	X	X	X
100%	[.673].195	X	X	X	X	X

The allocation to bonds is shown by the column headings. The allocation to equity is shown by the row headings. The allocation to the risk-free asset is the residual, if any. For example, in Panel 2A: Female, an allocation of 60% bonds, 20% equity and 20% risk-free assets yields an EPoR of .996 and an LPoR of .371.

Notation used in the paper.

- $(x) :=$  Current age.
- $\mu :=$  The drift of the diffusion process.
- $\sigma :=$  The volatility of the diffusion process.
- $V_t :=$  The investment process.
- $W_t :=$  The wealth process.
- $k :=$  The self-annuitization (consumption) rate.
- $X := \int_0^t (V_s)^{-1} ds$  : The Stochastic Present Value (SPV) of consumption.
- $M_i := E[X^i], i = 1, 2, 3, 4$ . The first four (non-central) moments of the SPV.
- $\mathbf{G}_R(x | \alpha, \beta) :=$  The cdf of the Reciprocal Gamma distribution
- $\mathbf{G}(x | \alpha, \beta) :=$  The cdf of the Gamma distribution
- $\alpha, \beta :=$  The two free parameters of the Reciprocal Gamma distribution.
- $\hat{\alpha}, \hat{\beta} :=$  The RG parameters that are ‘fitted’ to  $M_1$  and  $M_2$ .
- $\mathbf{J}(x | a, b, c, d) :=$  The cdf of the (Type II) Johnson distribution
- $a, b, c, d :=$  The four free parameters of the Johnson distribution.
- $({}_t p_x) :=$  The probability that age  $(x)$  survives to age  $t$ .
- $(m, l) :=$  The mode and scale parameters for the Gompertz distribution.
- $A(\xi | m, l, x) :=$  The Gompertz price of a life annuity, under a rate of  $\xi$ .
- $r :=$  The ‘real’ interest rate used in the ‘case example’.
- $\mu_e, \mu_b :=$  The drift coefficient for equity and bonds used in the ‘case example’.
- $\sigma_e, \sigma_b :=$  The volatility coefficient for equity and bonds used in the ‘case example’.

**Appendix A.** Proof of Theorem #2.

Equation (4) contains two parts, an exponential function,  $V_s$ , which is strictly greater than zero, multiplied by the term in “square brackets”, whose sign is indeterminate. Therefore, the process  $W_t$ , will be less than or equal to zero (ruin) at some future time  $T$ , *if and only if* the term in “square brackets” is less than or equal to zero. In other words,

$$W_T \leq 0 \quad \iff \quad \frac{w}{c} \leq \int_0^T (V_s)^{-1} ds. \quad (30)$$

On the other hand, the integral  $\int_0^T (V_s)^{-1} ds$  is monotonically non-decreasing with respect to the upper bound of integration  $T$ . This means that once  $\int_0^T (V_s)^{-1} ds$  becomes greater than  $w/c$ , it *stays* greater than  $w/c$ . Consequently, we arrive at our result that the probability  $W_t$  crosses zero, prior to some time  $T$ , is equivalent to the probability that  $W_T \leq 0$ . **Q.E.D.**

**Appendix B.**

The ‘fitted’ parameters that match the first two moments of the Johnson distribution are available explicitly. They are:

$$c = M_1 + d \exp\left\{\frac{1}{2b^2}\right\} \sinh\left\{\frac{a}{b}\right\} \quad (31)$$

and

$$d = \frac{M_2 - M_1^2}{\sqrt{\frac{1}{2} \left( \exp\left\{\frac{2}{b^2}\right\} \cosh\left\{\frac{2a}{b}\right\} - 1 \right) - \exp\left\{\frac{1}{b^2}\right\} \sinh^2\left\{\frac{a}{b}\right\}}} \quad (32)$$

Stated differently, using the parameters  $c$  and  $d$  defined by equation (31) and (32), the first two moments are assured to be matched. The parameters  $a$  and  $b$  must then be located by ‘solving’ for equality in the third and fourth moments.

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