Optimal Annuitization Policies: Analysis of the Options

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Abstract

At, or about, the age of retirement, most individuals must decide what additional fraction of their marketable wealth, if any, should be annuitized. Annuitization means purchasing a nonrefundable life annuity from an insurance company, which then guarantees a lifelong consumption stream that cannot be outlived. The decision of whether or not to annuitize additional liquid assets is a difficult one, since it is clearly irreversible and can prove costly in hindsight. Obviously, for a large group of people, the bulk of financial wealth is forcefully annuitized, for example, company pensions and social security. For others, especially as it pertains to personal pension plans, such as 401(k), 403(b), and IRA plans as well as variable annuity contracts, there is much discretion in the matter.

The purpose of this paper is to focus on the question of *when and if* to annuitize. Specifically, my objective is to provide practical advice aimed at individual retirees and their advisors. My main conclusions are as follows:

- Annuitization of assets provides unique and valuable longevity insurance and should be actively encouraged at higher ages. Standard microeconomic utility-based arguments indicate that consumers would be willing to pay a substantial "loading" in order to gain access to a life annuity.
- The large adverse selection costs associated with life annuities, which range from 10% to 20%, might serve as a strong deterrent to full annuitization.
- Retirees with a (strong) bequest motive might be inclined to self-annuitize during the early stages of retirement. Indeed, it appears that most individuals—faced with expensive annuity products—can effectively "beat" the rate of return from a fixed immediate annuity until age 75–80. I call this strategy *consume term and invest the difference.*
- Variable immediate annuities (VIAs) combine equity market participation together with longevity insurance. This financial product is currently underutilized (and not available in certain jurisdictions) and can only grow in popularity.

"I advise you to go on living solely to enrage those who are paying your annuities. It is the only pleasure I have left." — Voltaire

1. Introduction and Objectives

At some point during the retirement years, one must decide whether to annuitize any discretionary liquid savings. The process of annuitization involves purchasing a life annuity by paying a nonrefundable lump sum to an insurance company in exchange for a lifelong consumption stream that cannot be outlived. This decision is quite difficult since, on the one hand, the annuity will provide lifelong income. On the other hand, there is a serious loss of liquidity that comes with annuitization. This decision is faced by most individuals who are invested in variable annuity contracts—with the option to annuitize—as well as in 401(k), 403(b), IRA, and other personal pension plans.

This paper presents a dual, and perhaps conflicting, message. On the one hand, it argues that vol-

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untary annuitization provides invaluable longevity insurance that cannot be replicated using other investment vehicles. The longevity insurance guarantees that survivors will never run out of money, no matter how long they live. However, in contrast to the invaluable protection, it is an empirical fact that most consumers are reluctant to purchase life annuities actively. Rather, they prefer to create their own consumption stream—also known as self-annuitization—even at the expense of potential reductions in their standard of living.

Indeed, Modigliani (1986), Friedman and Warshawsky (1990), Mirrer (1994), and many other academic studies have documented that very few people consciously choose to annuitize discretionary wealth. This phenomena is especially puzzling within the paradigm of the Ando and Modigliani (1963) life cycle hypothesis (LCH), or Yaari (1965), under which individuals would seek to smooth their lifetime consumption by annuitizing wealth. Life annuities can "smooth" and "guarantee" consumption for the rest of one's natural life. The most common explanation for the thin annuity market "puzzle" is simply to abandon the strict form of the life cycle hypothesis and declare that individuals have strong bequest motives, as Bernheim (1991), Hurd (1989), and others have argued. Consumers with strong bequest motives are reluctant to annuitize since, in exchange for longevity insurance, there is little residual value left for the estate.

Another attempt to resolve the "low annuitization" puzzle is to argue that *even* when individuals have negligible bequest motives, annuities are simply *too expensive.* This line of thinking was introduced by Warshawsky (1988) and Friedman and Warshawsky (1990), and recently expanded by Mitchell et al. (1999). They show that the implied rates of return from life annuities are much lower, as a result of transaction costs or "loads," than are those available from other investment assets. (Although they also indicate that these loads have come down over time.) These loads may be partially attributable to the adverse selection implicit in the mortality tables. Nevertheless, they act to reduce the returns compared to other nonannuity alternatives. Other explanations have focused on the individual's ability to pool mortality risk in large families, the lack of real (inflation protected) annuities, and nonrational behavioral justifications.

In contrast to the academic literature that tries to explain or document the "thin" annuity market, the objective of this paper is to (1) demonstrate the important function that life annuities provide, using a simple microeconomic consumer choice model, and (2) then help retiring individuals decide if and when to purchase (additional) life annuities. The normative advice is provided by focusing on the *probability of consumption shortfall* as the operative measure risk. This paper is closely related to the recent work in Milevsky (1998), where the Canadian annuity market is analyzed in great detail vis-à-vis the probability of beating the return from a life annuity. The main idea behind the shortfall approach is to compute the probability of "beating" the rate of return from a life annuity. The higher this probability, the more it makes sense to wait before annuitizing, especially if there is a bequest motive. The reader is encouraged to consult Milevsky (1998) for more details on the simulation methodology as well as the parameter estimates from a cross-section of annuity prices.

This "probability-based" methodology allows one to quantify the opinion shared by most financial planners, namely, that consumers under age 75–80 should refrain from annuitizing any additional marketable wealth. The exception to this rule would be in the event that interest rates are extraordinarily high (cheap annuities) or when the consumer has private information that would lead him or her to believe that he or she is much healthier than the general population. The reluctance to annuitize is further reinforced by the inability of the consumer to acquire (at a reasonable cost) *real* indexed annuities that protect consumption against inflation, something that (arguably) equity markets are able to do quite effectively over long horizons.

I must, however, make absolutely clear that one of the main factors driving this result is the "spread" between the interest rate credited to the life annuity, and the rate available in the nonannuitized open market. As one can see from the actuarial model, in the event that this "profit spread" is zero, the probability of beating the life annuity is greatly reduced.

The remainder of this paper is organized as follows. Section 2 examines the theoretical benefits from annuitization in terms of the longevity insurance, mortality credits, and utility welfare improvements. Section 3 looks at the question of self-annuitization and the probability of being able to replicate a life annuity stream. Section 4 concludes the paper.

2. WELFARE ANALYSIS: THE BENEFITS TO ANNUITIZING

Consider a simple two-period example that illustrates the gains in utility from having access to a life annuity market. Assume we have one dollar that must be consumed during the next two periods. The consumption, denoted by C_1 and C_2, takes place at the end of the period. There is a p_1 probability that the individual will survive to (consume at) the end of the first period, and a p_2 probability of surviving to (consuming at) the end of the second period. The periodic interest rate is denoted by R. The objective is to maximize the discounted utility of consumption. To that end, I postulate logarithmic preferences. In the absence of annuities, the objective function and budget constraints are given by

$$\max_{\{C_1,C_2\}} \mathrm{E}[U] = \frac{p_1}{1+\rho} \ln[C_1] + \frac{p_2}{(1+\rho)^2} \ln[C_2], \tag{1}$$

$$\text{st} \qquad 1 = \frac{C_1}{1+R} + \frac{C_2}{(1+R)^2}, \tag{2}$$

where ρ is the subjective discount rate. Clearly, this model does not incorporate any utility of bequest, since only the "live" states are given weight in the objective function. The solution to this consumption-investment problem is obtained by creating the Lagrangian[1]

$$\max_{\{C_1,C_2,\lambda\}} L = \frac{p_1}{1+\rho} \ln[C_1] + \frac{p_2}{(1+\rho)^2} \ln[C_2] + \lambda\left(1 - \frac{C_1}{1+R} - \frac{C_2}{(1+R)^2}\right). \tag{3}$$

The first-order condition is

$$\frac{\partial L}{\partial C_1} = \frac{p_1}{(1+\rho)C_1} - \frac{\lambda}{(1+R)} = 0,$$

$$\frac{\partial L}{\partial C_2} = \frac{p_2}{C_2(1+\rho)^2} - \frac{\lambda}{(1+R)^2} = 0,$$

$$\frac{\partial L}{\partial \lambda} = -\frac{C_1}{(1+R)} - \frac{C_2}{(1+R)^2} + 1 = 0. \tag{4}$$

Solving the system of three equations and three unknowns, I obtain the optimal values for the choice variables:

$$C_1^* = \frac{p_1(\rho R + R + \rho + 1)}{p_2 + p_1\rho + p_1},$$

$$C_2^* = \frac{p_2(1 + 2R + R^2)}{p_2 + p_1\rho + p_1}. \tag{5}$$

The optimal consumption, in the absence of annuities, is given by Equation (5). The ratio of consumption between period 1 and period 2 is $C_1^*/C_2^* = p_1(1+\rho)/p_2(1+R)$. When the subjective discount rate is equal to the interest rate ($\rho = R$), then $C_1^*/C_2^* = p_1/p_2$, which is the ratio of the survival probabilities and is strictly less than one. Stated differently, the individual consumes less at higher ages. In fact, this result can be generalized to a multiperiod setting. When life annuities are not available, rational utility maximizers are forced to consume less as they age, even though their time preference is equal to the market rate.

However, in the presence of an actuarially fair life annuity market, the budget constraint in Equation (2) must change to reflect the probability adjusted discount factor. This greatly expands the opportunity set for the consumer and will increase the utility.

The optimization problem is now

$$\max_{\{C_1,C_2\}} E[U] = \frac{p_1}{1+\rho} \ln[C_1] + \frac{p_2}{(1+\rho)^2} \ln[C_2],$$

$$\text{st} \qquad 1 = \frac{p_1 C_1}{1+R} + \frac{p_2 C_2}{(1+R)^2}. \tag{6}$$

The Lagrangian becomes

[1] Of course, in this simple two-period model, we do not need the Lagrangian since we can always convert the problem to one free variable with no constraints. But in the general N period problem, this is how one would proceed.

$$\max_{\{C_1,C_2,\lambda\}} L = \frac{p_1}{1+\rho}\ln[C_1] + \frac{p_2}{(1+\rho)^2}\ln[C_2] + \lambda\left(1 - \frac{p_1C_1}{(1+R)} - \frac{p_2C_2}{(1+R)^2}\right). \quad (7)$$

The first-order condition is

$$\frac{\partial L}{\partial C_1} = \frac{p_1}{C_1(1+\rho)} - \frac{\lambda p_1}{(1+R)} = 0,$$

$$\frac{\partial L}{\partial C_2} = \frac{p_2}{C_2(1+\rho)^2} - \frac{\lambda p_2}{(1+R)^2} = 0,$$

$$\frac{\partial L}{\partial \lambda} = -\frac{p_1C_1}{(1+R)} - \frac{p_2C_2}{(1+R)^2} + 1 = 0. \quad (8)$$

The optimal consumption is denoted by C_1^{**}, C_2^{**}, and is equal to

$$C_1^{**} = \frac{\rho R + R + \rho + 1}{p_2 + p_1\rho + p_1},$$

$$C_2^{**} = \frac{1 + 2R + R^2}{p_2 + p_1\rho + p_1}. \quad (9)$$

The important point to notice is that $C_1^{**} = C_1^*/p_1$ and $C_2^{**} = C_2^*/p_2$, which implies that the optimal consumption is greater in both periods, in the presence of life annuities. Specifically, at time 0, the individual would purchase a life annuity that pays C_1^{**} at time 1 and C_2^{**} at time 2. The present value of the two life annuities—as per the budget constraint—is one dollar. In this case, the ratio of consumption between period 1 and period 2 is $C_1^*/C_2^* = (1+\rho)/(1+R)$. When the subjective discount rate is equal to the interest rate ($\rho = R$), then $C_1^*/C_2^* = 1$, which is the "smoothing" effect of annuities, discussed above.

Here is an numerical example that should help illustrate the model. Let $R = \rho = 10\%$, and let $p_1 = 0.75$ and $p_2 = 0.40$. The individual has a 75% chance of surviving to the end of the first period, and a 40% chance of surviving to the end of the second period. Hence, according to Equation (5), the optimal consumption is $C_1^* = 0.741$ and $C_2^* = 0.395$ in the absence of annuities. The maximum utility is $EU^* = -0.5115$. However, in the presence of life annuities, the optimal consumption becomes $C_1^{**} = 0.987$ and $C_2^* = 0.987$ with a maximal utility of $EU^* = -0.01247$, which is clearly greater than the no-annuity case. To get a sense of the benefit from annuitizing, if one solves Equation (7) with a budget constraint equal to 0.61 instead of 1, the optimal annuitized consumption would be $C_1^{**} = 0.603$ and $C_2^{**} = 0.603$. In this case the maximal utility would be the same as with the no-annuity case. Stated differently, if one were to take away 0.39 from the individual, but give him or her access to a fairly priced life annuity, the utility would be the same. The model presented obviously abstracts from many of the real-world issues that affect the decision to annuitize. Nevertheless, I believe that the intuitive implications are worth the price in assumptions. Annuities allow individuals to consume more—than they could have otherwise—during their retirement years. In our model a person would be willing to forgo up to 39% of his or her initial wealth to gain access to a fair life annuity.

3. Consume Term and Invest the Difference

3.1. Discrete Time: Deterministic Investment Returns

Given the reluctance of individuals to annuitize their liquid wealth—despite their welfare-enhancing properties—in this section I examine a strategy that seems to offer the best of both worlds. Specifically, I describe a strategy that attempts to replicate the income from a life annuity by self-insuring. The self-insurance is implemented early in retirement, and then, if so desired, wealth can be annuitized at a later age. I call this strategy *consume term and invest the difference.* The similarity to the well-known adage of buy term and invest the difference will be made clear in the process.

I will start the analysis with an intuitive discrete time example. The pricing definition of a one-dollar-per-year single-premium fixed immediate life annuity (FIA) is

$$a_x = (1+l)\left(\sum_{i=1}^{\infty} \frac{{}_ip_x}{(1+R)^i}\right). \quad (10)$$

This annuity pays one dollar at the end of every year, for the rest of the annuitant's life. I further assume no refunds, no certain periods, and no survivor benefits. The symbol R denotes the appropriate rate of interest, which is used by the insurance company to discount cash flows. The quantity ${}_ip_x$ denotes the conditional probability that an individual age x will attain age $x + i$, where it is understood that ${}_jp_n = 0$ for a large enough value of j. The survival probabilities are taken from an annuity mortality table. The proportional insurance load l incorporates all expenses, taxes, commissions, and distribution fees—let alone profits—and is multiplied by the pure actuarial premium to arrive at a market price a_x. Practically speaking, the quantity l is on the order of magnitude of approximately 0.15. (Although competitive pressures seem to have reduced this in recent years.) Stated differently, the pure actuarial premium is "grossed up" by approximately 15% to arrive at a market premium. See the work by Mitchell et al. (1999) as well as Milevsky (1998) for a further discussion of l. I should emphasize that the actual magnitude of the parameters will have a large effect on the optimal time—if any—to annuitize. Clearly, the larger the load l, the lower are the welfare gains to annuitization. In fact, if l is large enough, even in the absence of bequest motives, the optimal strategy is to avoid annuitization.

Now, let us see what happens if the retiree decides *not* to purchase the life annuity, but rather invest the funds in a liquid (nonannuitized) account and consume the same dollar as the annuity *would* have provided. Specifically, assume that the retiree, age x, decides to wait for one year and purchase the same annuity at age $x + 1$. In order to afford the exact same life annuity stream in one year, the annual investment return, denoted by K, earned by the retiree must be such that

$$a_x(1 + K) - 1 \geq a_{x+1}. \tag{11}$$

In other words, the life annuity premium at age x invested at a rate K, minus the one dollar consumption at the end of the year, must be greater than or equal to the market price of the annuity at age $x + 1$. Rearranging Equation (11) in terms of the investment return K, the condition for beating the rate of return from the annuity, over one year, is

$$K \geq K^* = \frac{a_{x+1}}{a_x} + \frac{1}{a_x} - 1. \tag{12}$$

I refer to K^* as the threshold annual investment return necessary for a successful deferral. In general, using the actuarial identity $({}_ip_{x+n}) = ({}_{n+i}p_x)/({}_np_x)$, I can rewrite a_{x+1} in terms of a_x, using Equation (10), and then rewrite the condition for beating the rate of return on the annuity, using Equation (12), as

$$K \geq K^* = \frac{1 + R}{{}_1p_x} - \frac{l}{a_x} - 1. \tag{13}$$

Equation (13) contains the main idea. It specifies the precise rate of return that a retiree must earn in order to "beat" the mortality-adjusted return from a life annuity. So long as the individual can earn at least K^*, it makes sense to self-annuitize and defer the decision until the next period. For example, when the insurance loads in Equation (13) are set equal to zero, the condition for beating the annuity is simply $K \geq K^* = (1+R)/({}_1p_x) - 1$. Since the term $({}_1p_x)$ is strictly less than one, the threshold return on investment K^*—in the no-load case—must be greater than the rate R. In actuarial terms, the term $({}_1p_x)^{-1}$ is referred to as "mortality credits" because they enhance the return R. The lower the probability of survival, the higher the mortality credits. Also, in general, a higher insurance load tends to reduce the threshold rate K^*. Of particular interest is the fact that Equation (13) indicates that, for a young enough individual x and a high enough insurance load l, the investment return threshold K^* could, in theory, be lower than R. In this case, one can beat the mortality-adjusted return from the life annuity by simply investing in the exact same assets used by the insurance company to discount cash flows.

In sum, if consumers can earn a (risk-free) return of K^*, they can defer annuitization by one period, yet consume the exact same amount that a life annuity would have provided.

3.2. Continuous Time: Deterministic Investment Returns

With the main idea behind us, I now move to a multiperiod analysis. My intention, once again, is to estimate the return required to beat the mortality-adjusted return from a life annuity. For the remainder of the paper I will consider life annuities that "pay out"—and are priced—in continuous time. To this end, one can imagine an immediate life annuity with a daily payout, although, obviously, the monthly variety is the most common. The continuity assumption, which is grounded in modern financial economic theory, simplifies the ensuing mathematics.

Using continuous compounding, the market price of a one-dollar-per-year life annuity for an individual at age x is

$$a_x = (1 + l) \int_0^\infty e^{-rt} {}_tp_x \, dt. \tag{14}$$

This time r denotes the continuously compounded interest rate, ${}_tp_x$ is the conditional probability that an individual age x survives to age $x + t$, and, once again, l denotes the insurance load charged. I would like to stress at this point that the "loading" on the actuarial premium is separate and distinct from the loading of any mortality tables. As such, the probability ${}_tp_x$ may include its own loading as well.

Following in the footsteps of recent work on annuity pricing by Frees, Carriere, and Valdez (1996), I adopt a model of mortality using a two-parameter Gompertz specification.[2] According to the mortality law proposed by Benjamin Gompertz, the force of mortality at any age x, is

$$\lambda(x) = \frac{1}{b} \exp\left(\frac{x - m}{b}\right). \tag{15}$$

The parameter m can be thought of as a modal lifetime, with b as a scaling variable. In this paper, I priced all annuities using the Individual Annuity Mortality (IAM) 2000 table, dynamically adjusted using scale G, published by the Society of Actuaries. Furthermore, the data were smoothed using the above-mentioned Gompertz specification. The exact parameters I used throughout the paper are $m = 88.18$ and $b = 10.5$ for males, with a life expectancy of 84.86 years, and $m = 92.63$ and $b = 8.78$ for females, with a life expectancy of 88.51 years. Table 1 provides the survival probabilities for a variety of ages.

Solving the integral in Equation (14), with mortality defined by Equation (28), I obtained a closed form (tractable) expression for the price of the life annuity:

$$a_x = (1 + l) \frac{b\Gamma(-br, b\lambda(x))}{\exp\{(m - x)r - b\lambda(x)\}}, \tag{16}$$

where $\Gamma(a, b)$ is the incomplete gamma function, defined to be $\Gamma(a, b) = \int_b^\infty e^{-t} t^{(a-1)} \, dt$, and is available in most mathematical software and spreadsheet packages. Despite the somewhat complex-looking expression in Equation (16), its closed-form representation allows us to price annuities using a spreadsheet and the appropriate values for r, l, m, b. Table 2 provides some annuity values for the above-mentioned male and female survival functions, assuming a "load" of $l = 0.10$. Recall that the annuity price is the "cost" of obtaining one dollar per annum for life.

Notice that the price of the annuity declines with both age and the interest rate. This observation is critical to our analysis. For example, if at age 65 interest rates are 8%, and they subsequently decline to 4% in 10 years, the price of the annuity will actually be more expensive at age 75.

Table 1

Survival Probability: Using Gompertz Fit to IAM 2000 Plus Scale G

Survive to Age:[a]	Male[b]	Female[c]
70	0.935	0.967
75	0.839	0.912
80	0.705	0.823
85	0.533	0.686
90	0.339	0.497
95	0.164	0.281
100	0.023	0.103

[a] Currently age 65.
[b] $\lambda(x) = \exp((x - 88.18)/10.5)/10.5$.
[c] $\lambda(x) = \exp((x - 92.63)/8.78)/8.78$.

Table 2

Sample Annuity Prices: Gompertz Mortality

Interest Rate (r)	Male		Female	
	a_{65}	a_{75}	a_{65}	a_{75}
0.04	14.426	10.569	16.184	12.127
0.05	13.121	9.848	14.583	11.216
0.06	11.999	9.206	13.222	10.410
0.07	11.027	8.630	12.058	9.693
0.08	10.180	8.112	11.054	9.055

Note. Force of mortality: male: $\exp((x - 88.18)/10.5)/10.5$, female: $\exp((x - 92.63)/8.78)/8.78$.

[2] One can "fit" the Gompertz distribution to any standard annuity table, to within 0.25% deviation in probability of death. Arguably, this approximation is good enough for our purposes, given the uncertainty we face in future investment returns and the analytic tractability of the Gompertz function.

Likewise in the other direction, if interest rates are currently "low" and they move up over time, the price of the annuity will decline with age, for two reasons.

I now return to my main objective, which is to examine the implications of deferring the decision to annuitize. In this model an individual with marketable wealth $W_0 = w$ at age x can choose to annuitize all liquid wealth. At a rate of a_x, per annual dollar of lifetime consumption, w can "buy" $c = w/a_x$ dollars per year of lifetime. Alternatively, the individual can invest the w in a portfolio, earning a (continuously compounded) rate of return k, and consuming the exact same (life annuity) amount c until the individual runs out of money at some future time t^*, which may be infinite. By construction, the investors wealth will obey an ordinary differential equation,

$$W_t = \begin{cases} \left(w - \dfrac{c}{k}\right)e^{kt} + \dfrac{c}{k}, & \text{for all } t < t^* \\ 0, & \text{for all } t \geq t^*. \end{cases} \tag{17}$$

Intuitively, the constant multiplying the e^{kt} in Equation (17) will be negative whenever the annuity payment c is greater than the perpetuity consumption defined by wk. The "negativity" forces the exponential term to overpower $+c/k$, and W_t will eventually hit zero. Understandably, if the return k is high enough, in other words, $w > c/k$, then $t^* = \infty$, and one can consume forever. Solving for t^*, in terms of the investment return k in Equation (17), and then substituting $a_x = w/c$, I obtain

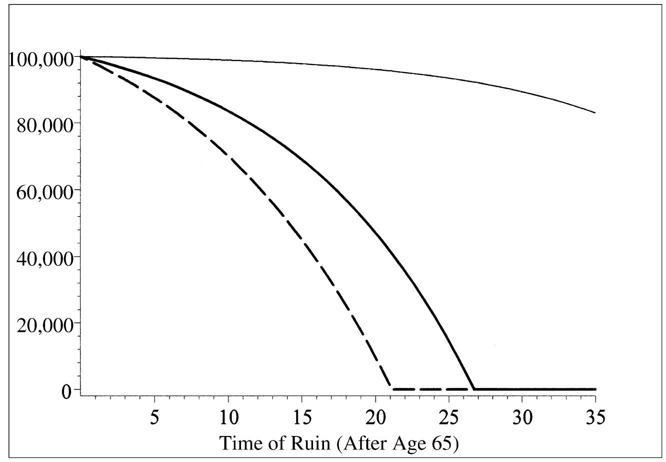
$$t^* = \begin{cases} \dfrac{-\ln\,[1 - a_x k]}{k}, & \text{for all } k < (a_x)^{-1} \\ \infty, & \text{for all } k \geq (a_x)^{-1}. \end{cases} \tag{18}$$

As we can see, when $k \geq (a_x)^{-1}$, the investor can safely beat the annuity forever, since $t^* = \infty$. In contrast, when $k < (a_x)^{-1}$, ruin (or shortfall) is certain, conditional on being alive. The lifetime probability of consumption "shortfall" is the probability of surviving to time t^*, which, as per Equation (28), is ${}_{t^*}p_x$.

Figure 1 illustrates the dynamic evolution of net wealth, as per Equation (17), using three different values for the parameter k. When $w = \$100{,}000$ with $l = 0.10$ and $r = 0.07$, then $a_{65} = 11.027$ (male) and the consumption rate is

Figure 1

Evolution of Wealth, as Function of Investment Rate *k* (7%, 8%, 9%), when Consumption Is Equal to Annuity Rate



$c = w/a_{65} = \$9{,}068$ per annum. If the individual decides to consume the \$9,068, while investing at the (same) $k = 0.07$ rate, then ruin will occur at time $t^* = 21.113$ years. There is a 0.49 chance of being alive at that point, which implies a 0.49 probability of ruin using this strategy. This is to be expected given identical rate of investment. On the other hand, if the investment rate is $k = 0.08$, then $t^* = 26.73$. Finally, when $k = 0.09$, $t^* = 54.262$, and the individual is "set" for life.[3]

The next question of interest becomes: At what time s will the marketable wealth from Equation (17) be equal to ca_{x+s}, the price of a continued lifetime consumption stream c? This will be the point at which the individual should "switch" and annuitize wealth. In other words, for the first few years, the consumer can earn more than the mortality-adjusted return. Eventually, the "mortality credits" are so large that it becomes optimal to annuitize.

Mathematically, I am searching for the (waiting period) value of s, as an implicit function of the investment return k, that satisfies

[3] Of course, this discussion assumes that the rate of return from the investment portfolio k is constant. When k itself is random, the probability of ruin will depend on the asset allocation within the portfolio, vis-à-vis the volatility of returns. See the related paper by Milevsky and Robinson (2000) for an analytic approximation to this "ruin probability."

$$\max_{0 \le s \le \infty} \{s\}$$

$$\text{st} \quad \frac{W_s}{a_{x+s}} \ge c. \tag{19}$$

Equation (19) argues that the consumer should defer annuitization until the original consumption stream is no longer affordable in the annuity market. Using Equation (17) together with the obvious condition that s^* should occur prior to ruin t^*, the "optimal annuitization" problem can be solved to yield

$$s^* = \begin{cases} \dfrac{1}{k} \ln \left[\dfrac{1/k - a_{x+s^*}}{1/k - a_x} \right], & \text{for all } k < (a_x)^{-1} \\ \infty, & \text{for all } k \ge (a_x)^{-1}. \end{cases} \tag{20}$$

Although the critical variable s^* appears on both sides of Equation (20), solving for s^* is quite easy with the use of a spreadsheet when the future annuity prices a_{x+s^*} can be stated with certainty.

3.3. Continuous Time: Stochastic Investment Returns

Let me sum up the main point of the previous section. If the future prices of all life annuities a_x and future investment returns k are known with perfect certainty, the individual can *consume term and invest the difference,* with no risk. This is done by locating the point at which the mortality-adjusted returns can not be "beaten." In practice, of course, the decision to postpone the purchase of a life annuity—and the implicit formulation from the previous section—is confounded by three major sources of uncertainty. There are three possible things that can go wrong with the decision to consume term and invest the difference. They are: (1) stochastic *investment* returns, (2) stochastic *interest* rates, and (3) stochastic *mortality* rates. By stochastic investment returns, I mean there is a chance that the rate of return k from the portfolio will not live up to expectations. This, of course, will imply that the evolution of (nonannuitized) wealth will not obey the ordinary differential equation stipulated in Equation (17).

Stochastic interest rate implies that the term structure of interest rates applicable in the market—and used by the insurance company to price annuities—fluctuates over time. Thus, once again, the price of the same exact annuity in 5, 10, 15, or 20 years is uncertain. Finally, even without the randomness in the discount factor, I do not know exactly what mortality table the insurance company will use when pricing the annuity in 5, 10, 15, or 20 years. This is what is meant by stochastic mortality.

Therefore, in practice, I do not know with certainty whether the investor will have enough money to purchase the exact same annuity in the future.

To quantify the risk of this strategy, I constructed a Monte Carlo simulation that generates thousands of future investment and interest rate scenarios. Each of these scenarios gave rise to a probability of a successful deferral. I will now explain in detail the exact method by which the randomness was generated.

3.3.1. Model for Investment Returns

I model continuously compounded investment returns, during any period in time, as normally distributed. This assumption is standard in financial economics and can be traced back to Boyle (1976) in the actuarial, risk, and insurance literature, as well as Black and Scholes (1973). Consequently, in sharp contrast to the deterministic Equation (17), the investor's portfolio will obey a geometric Brownian motion. The parameter μ will denote the growth rate of the portfolio (akin to k in the deterministic case), and the parameter σ will denote the volatility.

3.3.2. Model for Interest Rates

Similar to the model for investment returns, I assumed that the interest rate used by the insurance company to price annuities (or the valuation rate) obeys a mean reverting stochastic process. The process will have three free parameters. The first is $\bar{r}$, which denotes the long-run average level of the interest rate. The second parameter is γ, which denotes the speed of adjustment in the mean reverting prices. The final free parameter is σ_r, which denotes the volatility of interest rates. This continuous-time model of interest rate behavior was originally introduced by Cox, Ingersoll, and Ross (1985) and has been applied widely in financial economics. See Chan et al. (1992) for a discussion of the empirical estimates. The important point to note is that our simulations will allow for future random interest rates, but the randomness will be controlled by forcing interest rates to revert to a long-term level. If current rates are lower than the long-term rate $\bar{r}$, interest rates will be expected to increase. If, on the other hand, current rates are higher than the long-term rate $\bar{r}$, interest rates will be expected to decline. The rate at which the process moves back (reverts) to the long-term value is controlled by γ.

3.3.3. Model for Future Mortality Rates

One of the weak points in this kind of simulation analysis is that I do not know with certainty what particular mortality table the insurance company will be using in the future. Furthermore, if, as is the current trend, future mortality patterns continue to improve, annuity prices can only increase—even for a fixed interest rate.

To partially account for the problems in projecting future mortality trends, I have computed all annuity prices by dynamically projecting the Society of Actuaries Individual Annuity Mortality 2000 Table, using 100% of the scale G improvement factor. Essentially, this assumption implies that mortality *will* improve in time, but only as expected by scale G. If, indeed, future mortality improves by more than expected, our annuity prices will be too low. However, I note that the IAM 2000 was essentially constructed by projecting ahead the IAM 1983 table. So our methodology is consistent with the practice of updating mortality tables on a periodic basis. Technically speaking, our simulation model generated an interest rate for the deferral period in question and then priced the annuity using the IAM 20XX that would be applicable at that time.[4]

Finally, it is important to note that the individual *does not* have to estimate his or her own subjective mortality rate. Our simulation provides the probability of beating the life annuity, conditional on survival.

3.4. Description of the Monte Carlo Simulation

With full uncertainty in the model, I can compute the *probability* of a successful deferral. The actual probability I am looking for can be written as

$$\Pr\left[\frac{\tilde{W}_s}{\tilde{a}_{x+s}} \geq c\right]. \tag{21}$$

The crucial item, then, is to compute the distribution of the stochastic process $\tilde{c}(s) := \tilde{W}_s/\tilde{a}_{x+s}$, which is the consumption attainable, at time s, and then compute the probability that $\tilde{c}(s) \leq c$, the original consumption level.

I performed Monte Carlo simulations to obtain an empirical density function for values of $s = 5$, 10, 15, *and* 20 years. In particular, for each simulation run, the algorithm generated a vector of 25,000 random numbers for $\tilde{W}_s$, and a vector of 25,000 random numbers from $\tilde{a}_{x+s}$. The procedure then took the element-by-element ratio of the two vectors to obtain 25,000 random samples from the density function $\tilde{c}(s)$.

The program then counted the number of elements in the random sample that were less than the original consumption level c, thus estimating $\Pr[\tilde{c}(s) \leq c]$. I assumed that the future evolution of interest rates will obey the interest rate dynamics with parameters $\bar{r} = 0.085$, $\gamma = 0.25$, and $\sigma_r = 0.08$. Projecting ahead, I compute the relevant interest rate and apply the relevant mortality table with a load of $l = 10\%$ to obtain an estimate for the future annuity price. Likewise, I assume that the initial wealth is invested and consumed as per the geometric Brownian motion with parameters $\mu = 0.13$, $\sigma = 0.17$, as per the Ibbotson figures for the return on a well-diversified investment portfolio.

Here is a sample run. A 65-year-old male, in the year 2000, with \$100,000 in initial wealth, is contemplating buying a life annuity. The insurance company provides him with a quote of $a_{65} = \$11.027$ per dollar of lifetime consumption. This translates into an annuitized consumption of $c = 100{,}000/11.027 = \$9{,}068$ per year, which cannot be outlived. Now, if the 65-year-old decides to defer annuitization while investing in—and consuming from—a well-balanced equity portfolio, the simulations indicate the following: If he waits for 10 years, there is a 80% chance that he will be able to purchase the same exact annuity, or better, with the remaining funds.

Moreover, the benefits of waiting are numerous. The 65-year-old maintains liquidity and gains the utility of bequest as well as the possibility of an even greater annuity payment in the future.

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4 For related research on how to price the uncertainty surrounding future mortality rates, see Milevsky and Promislow (2000).

After conducting a large variety of simulation runs, it appears that the probability of a successful deferral is most sensitive to (1) the current level of interest rates in the market vis-à-vis the risk premium, $\mu - r_0$, and (2) the annuity insurance load l. In contrast, the parameters of the interest rate process ($\bar{r}$, γ, σ_r) have very little influence on the probability of a successful deferral. I believe this to be a direct manifestation of a "plateau" effect. Either the individual will have many times the amount of money needed to purchase the same exact life annuity, $\tilde{W}_s/\tilde{a}_{x+s} \gg c$, or the individual will have very little funds with which to purchase the life annuity, $\tilde{W}_s/\tilde{a}_{x+s} \ll c$. Consequently, the uncertainty surrounding the *future* interest rates will have little effect on the probability of a successful deferral. It also appears that when interest rates are low ($r_0 = 5\%$)—in a mean reverting environment—the probability of a successful deferral is somewhat invariant to the composition of the investor's portfolio. See Milevsky (1998) for extended results using a variety of parameter values.

In conclusion, simulations indicate that, in the current interest rate environment, a 65-year-old female (male) has a 85% (80%) chance of being able to beat the rate of return from a life annuity until age 80. Recall that the above-mentioned probabilities are all conditional on survival; thus, they uniformly overestimate the unconditional probability of consumption shortfall. Table 3 provides some additional numerical estimates for the probability of beating a life annuity.

3.4.1. Variable Immediate Annuities

In some sense, comparing the performance of equity-based investments with fixed-income products is misleading. Clearly, the reason we are able to beat the rate of return from the life annuity is that we are investing in assets whose expected growth rate is higher than the return from (low-risk) money market products. So, in some sense, we can argue that these results are driven by the long-term propensity of equity-based investments to outperform fixed income products, independently of the annuity structure. However, note that the equity (risky) return must do more than just "beat" the return from fixed income products; it must also beat the mortality credits. And, indeed, our simulations indicate that this is possible up until ages 75–80.

Of course, for those who are reluctant to hold (risky) equity-based investments as a substitute for the life annuity, the odds of beating the mortality credits are slim. Furthermore, if the implicit fees in the fixed immediate annuity are close to zero, the probability of "beating" the life annuity with identical fixed income products is essentially zero as well. In other words, for the deferral strategy to make sense, we must have two preconditions. First, the fees must be "high" enough; second, the alternative portfolio must be tilted toward equity products.

One final note worth mentioning is that variable immediate annuities (VIAs)—which currently are not very popular or widely used in many countries—are likely to grow in stature as the current generation of equity investors realize that VIAs provide longevity insurance together with the highly cherished equity exposure. Indeed, in a related research paper by Charupat, Milevsky, and Tuenter (2000) we document the substantial welfare gains that result from introducing VIAs to the market for longevity insurance. Basically, the optimal allocation within a payout annuity contract is identical to that outside of a payout annuity.

Table 3
Simulation Results: Probability of "Beating" the Life Annuity

Wait (Years)	$r = 5\%$		$r = 7\%$		$r = 9\%$	
	Male	Female	Male	Female	Male	Female
5	79.7%	81.1%	77.1%	78.3%	73.4%	75.1%
10	83.4	86.3	79.5	81.6	74.2	76.3
15	85.3	89.2	81.3	84.1	75.9	77.8
20	85.8	90.1	83.9	85.3	77.3	79.2

Note: Assumptions: $\mu = 13\%$, $\sigma = 17\%$, $l = 10\%$.

4. Conclusion

Life annuities provide valuable longevity insurance that, arguably, is just as important as traditional life insurance. Obviously, very strong bequest motives combined with unfavorable product pricing can severely reduce the desirability and appeal of annuity-like products. Nevertheless, as I illustrated in Section 2, the benefits to eventual annuitization are overwhelming. With that in mind, this paper advocates a retirement strategy that might be called "do it yourself and then switch." People who are reluctant to annuitize might consider creating their own annuity, by consuming at the same rate, so that there is a high enough probability of being able to purchase a similar product later in retirement. This allows the individual to maintain full control of the funds and to participate in the long-term upward performance of equity markets, as convincingly argued by Jeremy Siegel in *Stocks for the Long Run* (1995).

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Technical Appendix

In this paper I assume that the time-at-death random variable, $\tilde{T}$, can be expressed in a continuous-time manner. For an individual currently age x, the probability of death prior to time $t \geq 0$ (i.e., prior to *age* $x + t$) is modeled as

$$\Pr(\tilde{T} \leq t|x) := 1 - ({}_tp_x) = 1 - \exp\left\{-\int_0^t \lambda(x+s)\, ds\right\}, \quad (22)$$

where $\lambda(s)$ is the "force of mortality" and ${}_tp_x$ is the conditional probability that an individual age x will survive to time t. The function $\lambda(s)$ can heuristically be described as the *instantaneous* probability of death, applicable at time s. Although this may not be true in general, especially for low ages, I assume that $\lambda(s)$ is a strictly positive and increasing function; that is, $\lambda(s) > 0$ and $\lambda'(s) \geq 0$. Therefore, the function ${}_sp_x$ is monotonically decreasing in s. For example, the conditional probability that an x-year-old individual will survive to age 60 is clearly greater than the probability of surviving to age 61. Also, by definition, ${}_0p_x = 1$ and ${}_\infty p_x = 0$. See the classic textbook by Bowers et al. (1986) for additional information on mortality functions.

Equation (22) should be interpreted as a proper cumulative distribution function and denoted by $F(t)$, provided that $\int_0^\infty F'(t|x)\, dt = \int_0^\infty f(t|x)\, dt = 1$, where $f(t|x)$ is the probability density function of the time-at-death random variable for an individual age x. This, of course, puts an additional restriction on the force of mortality $\lambda(s)$, namely,

$$F(\infty) = \int_0^\infty f(t|x)\, dt = \int_0^\infty \lambda(t) \exp\left\{-\int_0^t \lambda(s)\, ds\right\} dt = 1. \quad (23)$$

It follows, therefore, from Equation (22) that

$$\int_0^\infty ({}_tp_x)\lambda(t)\, dt = 1. \quad (24)$$

A simple application of the chain rule retrieves the convenient relationship:

$$\lambda(x+t) = \frac{f(t|x)}{1 - F(t|x)}. \quad (25)$$

Finally, the expected remaining lifetime in this framework is

$$E[\tilde{T}|x] = \int_0^\infty t\, f(t|x)\, dt : = \int_0^\infty t\, \lambda(x+t) \exp\left\{-\int_0^t \lambda(x+s)\, ds\right\} dt. \quad (26)$$

As a special case, when $\lambda(s)$ is constant and equal to λ for all ages and times, Equation (22) leads to $F(t|x) = 1 - e^{-\lambda t}$, and therefore $f(t|x) = \lambda e^{-\lambda t}$ and $E[\tilde{T}|x] = 1/\lambda$. In other words, a constant $\lambda(s) = \lambda$ implies an exponential distribution of time-at-death. In this case, an individual's expected *remaining* lifetime is equal to the reciprocal of the force of mortality, regardless of his or her current age. Although this particular form is quite convenient to work with, it obviously has the undesirable property that the probability of death is identical throughout the human life cycle.

A more realistic continuous-time force-of-mortality assumption is the Gompertz law. The exact specification of this distribution is

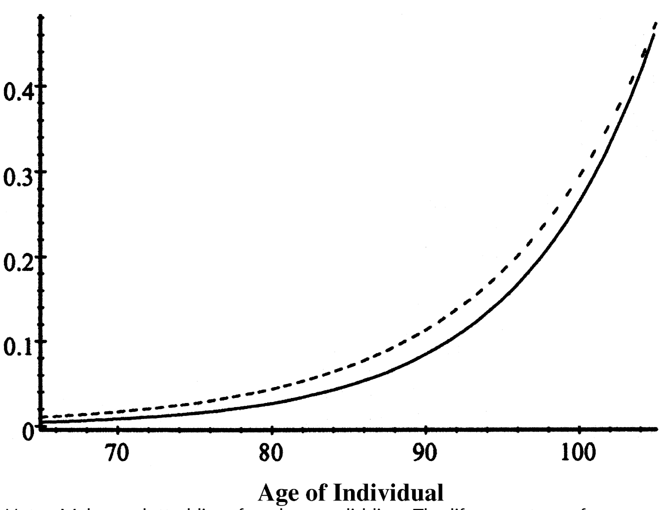
$$\lambda(x|m, b) = \frac{1}{b} \exp\left\{\frac{x - m}{b}\right\}. \tag{27}$$

Accordingly, as per Equation (22), the conditional probability of survival is

$$\begin{aligned} {}_tp_x &= \exp\left\{ -\int_0^t \lambda(x + s)\, ds \right\} \\ &= \frac{\exp(\exp\{-m/b\}(1 - \exp\{(x + t)/b\}))}{\exp(\exp\{-m/b\}(1 - \exp\{x/b\}))}. \end{aligned} \tag{28}$$

The parameter m is the mode, and the parameter b is the scale measure, of the probability distribution. The exact values of m and b clearly depend on the cohort in question, as well as the type of mortality table being modeled.

Figure 2
Force of Mortality Curve, Starting at Age 65, Using the Dynamically Adjusted IAM 2000 Table.



Notes: Males = dotted line, females = solid line. The life expectancy for males is 84.86 years and for females 88.51 years.

For illustrative purposes, Figure 2 displays the "force of mortality" curve—for males and females—taken from the IAM 2000 Table. On a technical note, the data were dynamically projected using scale G and then smoothed using the above-mentioned Gompertz specification. The exact parameters are $m = 88.18$ and $b = 10.5$ for males, and $m = 92.63$ and $b = 8.78$ for females. Clearly, the force of mortality is higher, at all ages, for males compared to females.

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