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Asset Allocation Via The Conditional First Exit Time or How To Avoid Outliving Your Money

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Abstract. The risk of outliving your money (or shortfall) with low risk, low return investments is very often more serious than the risk of losing money on high risk investments, until quite late in life. A stochastic process model incorporating mortality tables for men and women of retirement age, random rates of return and fixed initial wealth and desired level of consumption provides the analytical tool. A simulation using Canadian mortality tables and rates of return shows that almost all retirees should invest some of their wealth in equity, and for many the optimal allocation is 70–100% equity. The risk of shortfall is surprisingly high for a reasonable range of values of the variables, especially for an allocation of 100% in treasury bills. Women face much greater risk of shortfall than men. The analytical model also permits calculation of the distribution of the bequest and hence allows an individual to trade off changes in shortfall risk against changes in the expected bequest to the heirs.

Key words: asset allocation, shortfall risk, retirement planning

How should an individual who is retired allocate his or her investments between high risk, high return assets and low risk, low return assets? The retiree faces two risks. If s/he invests in low return assets, s/he risks outliving the income stream such an investment generates. If s/he invests in high return assets, there is a chance that losses will diminish the asset base and also lead to starvation.

The conventional wisdom has always been that investors should pick equities or other high return investments earlier in their life cycle, and gradually switch to bonds and treasury bills later in the life cycle. By retirement age, they should be holding more than half their investments in bonds and near-cash securities. For example, [Malkiel 1990] says that investors in their late sixties and older should hold 60% in bonds, 30% in equity and 10% in a money market fund. Investors in their mid-fifties are recommended to have 50% in stocks, 45% in bonds and 5% in the money market. Malkiel bases all his recommendations on the variability of returns, without explicitly considering either the expected lifespan of investors or the risk that they might outlive their money.

[Ho, Milevsky and Robinson 1994] find that retiring individuals would be better served by holding more of their portfolios in high risk assets. Their analytical formulation minimizes the probability of failing to reach a required rate of return each year by allocating assets between treasury bills and equity. They find that equity allocations far greater than those suggested by Malkiel are needed to minimize the chance of missing the required return. The return patterns over time noted by [Butler and Domian 1993] seem to support this result.

The investment allocation in this paper incorporates the required rate of return to minimize the probability of failing to meet that rate of return over the weighted lifespan remaining to the person. The objective is to minimize the probability of running out of money before the (uncertain) date of death. This implied utility function of minimizing shortfall is somewhat similar to the approach taken by [Leibowitz and Kogelman 1991]. They "measure risk by the "shortfall probability" relative to a minimum return threshold." A fund manager can choose any feasible combination of minimum return and probability and then allocate the assets between a risky and a risk-free asset to attain a desirable position. Their procedure does not endogenize the time horizon of the investor, since fund managers do not necessarily have a specific time constraint. They do observe that for longer time horizons, the proportion invested in equity rises.

Other researchers have considered the general question of which investment horizon to use and what effect different horizons have on how we view risk and return. In general, they find that the annualized standard deviation of riskier assets declines if they are held without trading for long periods. Different assets do well in specific shorter periods of time; so the benefits of changing portfolio composition are considerable if the investor times successfully.[2] The implication for asset allocation is that you should use more equity for longer horizons. [Lloyd and Modani 1983] conclude:

> In general, the usefulness of time diversification is more evident for portfolios containing common stock. Further, the riskiness of any portfolio position is unclear unless the number of time periods the portfolio will be held is also considered. (pp. 11)

Since we are solving the problem for an individual retiree, we incorporate this time dimension explicitly. In addition, we require annual consumption from the portfolio, which does not appear in other researchers' treatments of this problem. Substitution of standard Canadian mortality tables and reasonable estimates of return and variance for Canadian T-bills and equity provides surprising results. Virtually every retiree should include some equity in the retirement portfolio, even at as late an age as 75. For many people, very high allocations to equity are optimal, sometimes 100%. Even with the optimal portfolios, shortfall risk is quite high until late in life or with substantial wealth relative to consumption. Women face much higher risk than men.

The more common approach to asset allocation in the finance literature is to maximize expected utility subject to some constraints. This procedure involves assuming a utility function and an appropriate aversion to risk.

In using the shortfall approach, we adopt a utility function and a risk aversion parameter together. The utility function is the desire to consume at some level appropriate to the person's situation. The risk is the probability of not reaching the specified level of consumption. This is like using a binary zero-one utility function, similar to the practice in immunization.

In the rest of the paper we derive the formal mathematical model for optimizing the retiree's asset allocation. Since the derivation does not produce a closed form solution, we developed a Monte Carlo simulation. We present realistic numerical examples to illustrate the details, and then draw more general conclusions from a summary of many simulations. We use some alternative asset classes to illustrate the range of possible results. We then provide the expression for the probability distribution of the bequest function and present some numerical results. We conclude with a summary of the results, and suggestions for further research and improvements in our techniques.

1. Minimize the probability of outliving wealth

Our model assumes that at retirement ($t = 0$) the retiring-investor deposits all of his/her current wealth (W_0) into an account that allows him/her to allocate funds to and from various asset categories, within the account, at fixed points in time. In addition, the retiring-investor consumes from this account fixed sums at fixed points in time, as long as there is enough wealth to cover the withdrawal. Hence, the account is instantaneously aware of its own current market value and does not allow a withdrawal that exceeds its own net worth. Furthermore, when the retiring-investor consumes, the account dispenses funds from each category in a way that is proportional to the market value of funds in each category.

Consumption per year is fixed in this model. If a retired person earns unexpectedly high rates of return, then perhaps he/she will increase his/her consumption, and the reverse if low rates are realized. The point of our model, and the fixed consumption assumption, is to help the investor minimize the probability of earning a rate of return that will require a downward adjustment in consumption. Thus, a fixed level of desired consumption is the logical starting point.

First we analyze the probability of an individual outliving his/her wealth under one asset category with deterministic interest rates. We then generalize the model to a stochastic rate of return which depends on various asset categories and the respective asset allocation proportions in each.

We express everything in real dollars and real rates of return. The mathematics could be done in nominal rates and we would obtain the same results. The simplicity of real rates is the sole reason for doing it this way.

1.1. Deterministic rates of return

For notational simplicity, time t is measured in units of years and each year is subdivided into k non-overlapping periods of equal length $1/k$ years. Therefore, n periods will be synonymous with n/k years. Also, we use and refer to both an *effective annual* interest rate as well as to an *effective periodic* interest rate. From a purely mechanical point of view, we assume that the real annual consumption rate is C and hence at the end of each period the retiring-investor will consume C/k, furthermore we set the interest compounding periods to correspond with the withdrawal periods.

In the deterministic case, the effective annual interest rate is r. There are k compounding-withdrawal periods per year, therefore the effective the one period interest rate is $\sqrt[k]{1+r}-1$.

Hence, at the end of the first period, ($1/k$ years after retirement and immediately after the first withdrawal), the investor's wealth W_1 can be represented as:

$$W_1 \equiv \max\left\{0, \left(W_0 \times \sqrt[k]{1+r} - \frac{C}{k}\right)\right\}$$

The reason for the max{} term is that wealth cannot become negative; in other words there is no line of credit. This means that the retiring-investor cannot withdraw C/k dollars from an account whose market value is less than C/k dollars.

Likewise, at the end of the second period, ($2/k$ years after retirement and immediately after the second withdrawal), the investor's wealth W_2 can be represented as:

$$W_2 \equiv \max\left\{0, \left(\left(W_0 \times \sqrt[k]{1+r} - \frac{C}{k}\right) \times \sqrt[k]{1+r} - \frac{C}{k}\right)\right\}$$

In general, at the end of the n'th period, (n / k years after retirement and immediately after the n'th withdrawal), the investor's wealth W_n can be represented as:

$$W_n \equiv \max\left\{0, \left(\left(\ldots\left(\left(W_0 \times \sqrt[k]{1+r} - \frac{C}{k}\right) \times \sqrt[k]{1+r} - \frac{C}{k}\right)\ldots\right) \times \sqrt[k]{1+r} - \frac{C}{k}\right)\right\} \tag{1}$$

which after some algebraic manipulation can be expressed as:

$$W_n \equiv \max\left\{0, \left(W_0 \times (\sqrt[k]{1+r})^n - \frac{C}{k} \times \left(\sum_{i=0}^{n-1} (\sqrt[k]{1+r})^i\right)\right)\right\} \tag{2}$$

Moreover, by viewing the second term in equation (2) as the accumulated value of an annuity due, (or by adding the terms of a geometric series) one can re-write W_n as:

$$W_n \equiv \max\left\{0, W_0 \times (1+r)^{\frac{n}{k}} - \frac{C}{k} \times \left(\frac{(1+r)^{\frac{n}{k}} - 1}{\sqrt[k]{1+r} - 1}\right)\right\} \tag{3}$$

Now, as a result of the deterministic interest rate involved, one can calculate the exact period N^* when the investor's wealth W_{N^*} will equal zero for the first time.

$$W_0 \times (1 + r)^{\frac{n}{k}} = \frac{C}{k} \times \left(\frac{(1 + r)^{\frac{n}{k}} - 1}{\sqrt[k]{1 + r} - 1} \right) \tag{4}$$

After some elementary algebra N^* translates into the smallest integer greater than or equal to

$$\frac{\ln[C] - \ln[C - W_0 k((1 + r)^{1/k} - 1)]}{\frac{1}{k} \ln[1 + r]} \tag{5}$$

Strictly speaking, the only time the investor can run out of money is during a withdrawal, i.e. when there is not enough money in the account to satisfy the consumption requirement. Therefore N^* is defined as the integer-valued withdrawal period when the investor no longer is able to satisfy the entire consumption requirement. Thus, when interest rates are known and constant, the investor will run out of money at (the end of) period N^*; of course this is provided s/he is alive at (the end of) N^*.

However, by incorporating mortality functions one can strengthen the above statement by observing that, under deterministic interest rates, the probability that a retiring-investor outlives his or her money is:

$$\mathbf{P}(starve) = P^x_{N^*/k} \tag{6}$$

P^x_t denotes the probability that an individual aged x (in years) will survive to age $x + t$ (in years), taken from any standard mortality table. In words, equation (6) represents the probability that an individual who is alive at age x will survive to age $x + N^*/k$, which happens to be the time (in years) when the individual runs out of money. Hence the probability of starvation is the probability that the retiring-individual lives to the point where s/he runs out of money.[3]

In particular, (and extremely desirable) N^* may be infinite. This occurs when the (deterministic) interest rate r is large enough so as to establish an eternal perpetuity. Mathematically this occurs when the argument in the logarithmic expression of equation (5) becomes negative,[4] which is when:

$$r \geq \left(1 + \frac{C/W}{k}\right)^k - 1 \tag{7}$$

In which case $N^* = \infty$ and then:

$$\mathbf{P}(starve) = P^x_\infty = 0 \tag{8}$$

Under such circumstances, one will never starve to death because there is enough initial wealth to create a consumption perpetuity that will last for ever.

One final point about deterministic interest rates is to investigate when the frequency k of compounding-withdrawal approaches infinity, in which case, from equation (5), the appropriate *time* of starvation $T^* = N^*/k$, will be:

$$T^* = \lim_{k\to\infty} \frac{\ln[C] - \ln[C - W_0 k((1+r)^{1/k} - 1)]}{\ln[1+r]} = \frac{\ln[C] - \ln[C - W_0 \ln[1+r]]}{\ln[1+r]}$$

The proof of which can be obtained by noticing that for $a > 0$:

$$\lim_{k\to\infty} k(a^{1/k} - 1) = \ln[a]$$

Likewise, from equation (7), $T^* = \infty$ for continuous compounding-consumption, when:

$$r \geq \lim_{n\to\infty} \left(1 + \frac{C/W}{k}\right)^k - 1 = e^{C/W} - 1$$

Which gives us a practical *rule of thumb* formula for analyzing the investment-consumption problem (under deterministic interest rates) without resorting to a specific withdrawal frequency.

We are now ready to generalize the above discussion to stochastic rates of return.

1.2. Stochastic rates of return

Suppose there is a collection of m asset categories, each with its own stochastic behaviour, and let $\vec{\alpha}$ denote the m dimensional vector of asset allocation proportions. At retirement the retiring-investor specifies a particular $\vec{\alpha}$ which he/she would like to maintain. We assume that the retiring-investor chooses an $\vec{\alpha}$ and adheres to it throughout his/her remaining lifetime. This may sound like a very strict requirement, and in practice we would recommend that the investor *update* his/her $\vec{\alpha}$ after every withdrawal. However, for the purpose of the model, the static assumption is necessary for obtaining an estimate of the probability of outliving wealth. This information should act as a guide for the asset allocation decision now, even if the investor will almost surely change the proportions at the next withdrawal period.[5]

The stochastic scenario analogue of the deterministic one period interest rate $\sqrt[k]{1+r} - 1$ will be a random variable $\mathbf{R^k}(\vec{\alpha})$ which is an explicit function of the vector $\vec{\alpha}$ as well as the number k of compounding-withdrawal periods per year. In addition it is an implicit function of the underlying asset return specification which is a multivariate distribution denoted by Λ. In this paper we assume that all financial assets can be modeled as a Geometric Brownian Motion.[6]

Therefore, one year returns are Multivariate Lognormally distributed with parameters (μ, Σ) where μ is the one year mean vector of logarithmic returns and Σ is the one year variance-covariance matrix of logarithmic returns. Furthermore, using the properties of the Lognormal distribution [Crow and Shimizu 1985], one *period* returns are Multivariate Lognormally distributed with parameters (k/μ, [1/k] Σ) Furthermore, if $\vec{\alpha}$ is the vector of asset allocation proportions whose elements are α_i and if we let I_i denote an m dimensional row vector with a 1 in position i and 0 in all other positions, then the desired random variable $\mathbf{R^k}(\vec{\alpha})$ is:

$$\mathbf{R^k}(\vec{\alpha}) = \sum_{i=1}^{m} \alpha_i \cdot \Lambda\left(I_i \cdot \frac{1}{k}\mu, I_i \cdot \frac{1}{k}\Sigma \cdot I_i'\right) \tag{9}$$

The notation, $\mathbf{R^k}(\vec{\alpha})_i$, would represent a realization of the one period return random variable, in other words the *actual* return in the i'th period.

Continuing as in the deterministic case, at the end of the first period the investor's wealth W_1 is a random variable that can be represented as:

$$W_1 \equiv \max\left\{0, \left(W_0 \times \mathbf{R^k}(\vec{\alpha})_1 - \frac{C}{k}\right)\right\}$$

Likewise, at the end of the second period the investor's wealth W_2 is a random variable that can be represented as:

$$W_2 \equiv \max\left\{0, \left(\left(W_0 \times \mathbf{R^k}(\vec{\alpha})_1 - \frac{C}{k}\right) \times \mathbf{R^k}(\vec{\alpha})_2 - \frac{C}{k}\right)\right\}$$

In general, at the end of the n'th period, the investor's wealth W_n is a *random variable* that can be represented as:

$$W_n \equiv \max\left\{0, \left(\left(\ldots\left(\left(W_0 \times \mathbf{R^k}(\vec{\alpha})_1 - \frac{C}{k}\right) \times \mathbf{R^k}(\vec{\alpha})_2 - \frac{C}{k}\right) \times \ldots\right) \times \mathbf{R^k}(\vec{\alpha})_n - \frac{C}{k}\right)\right\} \tag{10}$$

which, after some algebraic manipulation, can be expressed as:

$$W_n \equiv \max\left\{0, \left(W_0 \times \prod_{i=1}^{n} \mathbf{R^k}(\vec{\alpha})_i - \frac{C}{k} \times \left(\sum_{i=z}^{n} \prod_{j=i}^{n} \mathbf{R^k}(\vec{\alpha})_j\right) - \frac{C}{k}\right)\right\} \tag{11}$$

As before, we are interested in the first time W_n reaches zero, denoted by N^*. However, since returns are stochastic, N^* is a random variable (otherwise known as a stopping time). Specifically:

$$N^* = \inf\{n \geq 0; W_n = 0\} \tag{12}$$

N^* is the First Exit Time of the stochastic process W_n from the set of non-zero numbers. Technically speaking, $P[N^* = i]$ is an implicit function of W_0, C, k, $\vec{\alpha}$, Λ which for obvious reasons satisfies:

$$\sum_{i=1}^{\infty} P[N^* = i] = 1 \tag{13}$$

A non-zero $P[N^* = \infty]$ denotes the probability that W_n never equals zero i.e. that the investor is set for an eternal life.

As in the deterministic case, we would like to calculate the probability of living to the time (period) when the money runs out. However, since we do not know the exact N^* when the wealth will first be zero, the best we can do is compute the probability that $N^* = i$ for all possible i, then compute the probability of surviving to age $x + i/k$, after which we multiply those two numbers and then finally add them up over all i. Mathematically, we obtain that the probability of outliving wealth[7] (under stochastic rates of return) is:

$$\mathbf{P}(starve) = \sum_{i=1}^{\infty} P^{x}_{i/k} \cdot P[N^* = i] \tag{14}$$

One should view $\mathbf{P}(starve)$ as the conditional (on being alive) First Exit Time of the stochastic process W_n from the set of positive real numbers.

The general objective of the remainder of this paper is to:

- Under a Lognormal distributional assumption for Λ together with a given k, W_0, C; compute the numerical value of $\mathbf{P}(starve)$ in equation (13) for various $\vec{\alpha}$ values.
- For the above conditions, find a suitable $\vec{\alpha}^*$ that will minimize $\mathbf{P}(starve)$ in equation (13) and hence minimize the probability of outliving wealth.

2. Numerical results

We first summarize our findings, to make it easier for the reader to follow the details:

1. Most retirees will materially reduce the probability of shortfall in their retirement income by investing part of their money in a high return, high risk asset like common equity.
2. The probability of shortfall at the optimal allocation is fairly high for all but those who are quite late in life or quite wealthy.
3. The probability of shortfall is much higher for a woman than for a man of the same age with the same wealth to consumption ratio. The optimal asset allocation doesn’t differ much between men and women, though the recommended equity holding is generally a bit higher for women.

4. Most of the reduction of shortfall risk comes in moving from no equity to about 40–50% equity in the portfolio. The function we are minimizing is quite flat in the vicinity of the optimal allocation; so a range of equity allocation percentages will give what is for practical purposes the same shortfall probability.
5. Replacing treasury bills with bonds gives much the same insights. Replacing the index equity portfolio with a small cap portfolio reduces shortfall risk materially and reduces the optimal allocation to equity somewhat.
6. The expected bequest increases monotonically with α. An individual could trade off a small increase in shortfall risk for a larger expected bequest by increasing α beyond the allocation that minimizes shortfall.

We cannot solve for the probability of shortfall as a function of asset allocation analytically. The authors developed a computer program, in Turbo C++, that estimates the magnitude of $\mathbf{P}(starve)$ in equation (13) for $k = 12$ (which is monthly withdrawals) and for various $\vec{\alpha}$ values of dimension $m = 2$ (which is two asset categories). We use a Monte Carlo Simulation of $P[N^* = i]$ for a Lognormal distribution, in conjunction with monthly Male/Female mortality rates estimated from yearly actuarial mortality tables. Since the simulation is attempting to represent an analytical curve, there will be some variation between different runs, and the values we obtain are only close approximations.

Initially, the two asset categories are Canadian equity and Canadian treasury bills, representing high risk, high return investments and low risk, low return investments, respectively. We must emphasize that in this section we are providing estimates of the parameters for the procedure. Any retiree or analyst could provide his or her own values, to determine appropriate asset allocations. We use long-run historical averages as reasonable estimates of future long-run returns. We assume a well-diversified portfolio of common equity[8], no transactions costs and no differential taxes.[9] The withdrawal amount is thus pre-tax annual income in retirement, in real dollars, although for convenience we continue to call it consumption.

We draw the mortality data from Statistics Canada Health Reports Supplement No. 13, (1990) Vol. 2, No. 4. We use historical real rates of return for 1950–93 from [Ho and Robinson 1996], Appendix D. The average real return on Canadian equity is estimated to be 7.48% per year, with a standard deviation of 16.82%. The average real return on Canadian treasury bills is estimated to be 1.85% percent with a standard deviation of 3.08%. Each simulation is run with 3500 trials.

We illustrate the results first with three examples encompassing reasonable situations (see Table 1).[10] Each example shows the probability of shortfall for a man or a woman, for increments of five percentage points of the wealth invested in equity. All the monetary amounts are in constant dollars at the date of retirement.

Example 1. The person retires at age 65 with wealth of \$560,000 and wishes to withdraw \$40,000 p.a. (i.e. $W/C = 14$). The optimal allocation to equity is 100% for a man or woman, and the probability of shortfall is 17.8% for the man and 28.2% for the woman. Note that the probability of shortfall is very high if they invest entirely in treasury bills—44% and 65%, respectively. Possibly they could reduce their risk of shortfall some-

Table 1. **Probabilities of Shortfall: Examples of Simulation Results.** Each entry is the probability the specified individual will suffer a shortfall in desired consumption given the percentage of wealth invested in common equity shown in the left-hand column. W/C is the wealth to consumption ratio in real $. The minimum probability of shortfall is in boldface italics.

% in	Age 65, W/C = 14		Age 75, W/C = 14		Age 65, W/C = 20	
Equity	Male	Female	Male	Female	Male	Female
0	.442	.654	.150	.289	.138	.292
5	.420	.622	.134	.262	.099	.240
10	.400	.594	.120	.239	.087	.202
15	.368	.563	.107	.219	.058	.166
20	.350	.553	.089	.195	.055	.132
25	.332	.531	.089	.186	.045	.112
30	.315	.516	.080	.176	.048	.092
35	.300	.455	.068	.148	.035	.093
40	.264	.452	.067	.152	.043	.087
45	.267	.428	.068	.130	.037	.070
50	.237	.399	.059	.124	.038	.078
55	.220	.369	.060	.124	.035	.081
60	.241	.370	.055	.124	***.027***	.078
65	.200	.345	.058	.115	.040	***.069***
70	.198	.341	.059	.128	.042	.072
75	.189	.322	.061	.105	.040	.082
80	.207	.295	.064	***.099***	.041	.078
85	***.178***	.302	.059	.117	.047	.088
90	.203	.300	.070	.119	.044	.079
95	.187	.283	***.054***	.125	.050	.090
100	***.178***	***.282***	.062	.116	.055	.089

what by borrowing and investing the proceeds in equity as well, but we do not consider that topic in this paper.[11]

Example 2. The person has the same wealth and required withdrawal as in the first example, but retires at age 75. Now the optimal allocation is an interior solution. A male should place 95% in equity, to face a shortfall risk of 5.4%. A female should place 80% in equity, with a shortfall risk of 9.9%. Note that while the shortfall probabilities decline and then rise again, they do not do so uniformly. The decline in shortfall probability caused by investing increasing amounts in equity from 0 to the optimum is much faster than the subsequent rise caused by investing more equity than the optimal amount. Furthermore, there is some unevenness. For example, the minimum for the female is reached at a shortfall probability of 9.9% (70% equity), but it then moves: 11.7% (85%); 11.9% (90%); 12.5% (95%); and 11.6% (100%). The simulation causes these anomalies, since the curve is very flat around the optimal point. Figure 1 shows this clearly for a selection of values of *W/C* and equity allocations.

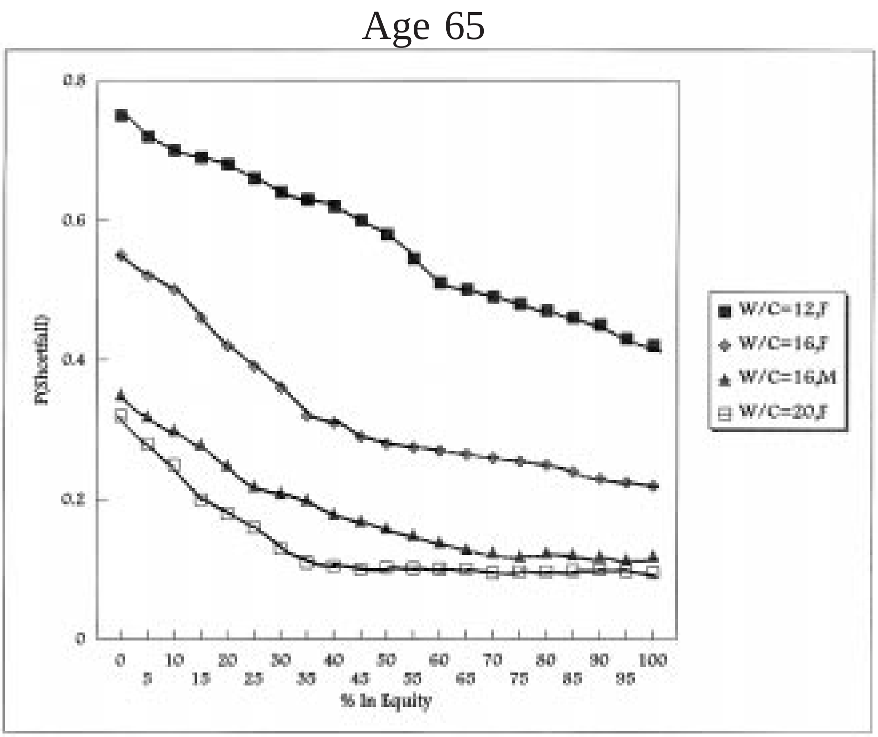

Figure 1. The relationship between the probability of shortfall and the percentage allocation of wealth to equity for females with wealth to consumption ratios of 12, 16 and 20 and a male with a wealth to consumption ratio of 16.

Example 3. The person retires at 65 with wealth of $540,000 and desires to have $27,000 per year retirement income (i.e. $W/C = 20$). The last two columns of Table 1 show that the shortfall probabilities have been reduced considerably when compared with the first example, with its higher required income. The optimal allocation to equity is still quite high, however, at 60% for a man and 65% for a woman.

2.1. Generalizing the results

Figures 1 and 2 show the pattern of allocations quite clearly for sets of different *W/C* levels for a single age and different ages for a single *W/C* level. For the youngest age and lower *W/C* combinations, the curve is fairly steep, and the risk of shortfall remains high even at 100% equity. For the rest, the curve is quite steep up to 40% or 50% equity, then becomes relatively flat. Thus, it makes little difference how retirement assets are allocated over large ranges, as long as a substantial portion is in equity. The risk of shortfall is very high for an allocation of 100% treasury bills in all but the most favourable situations.

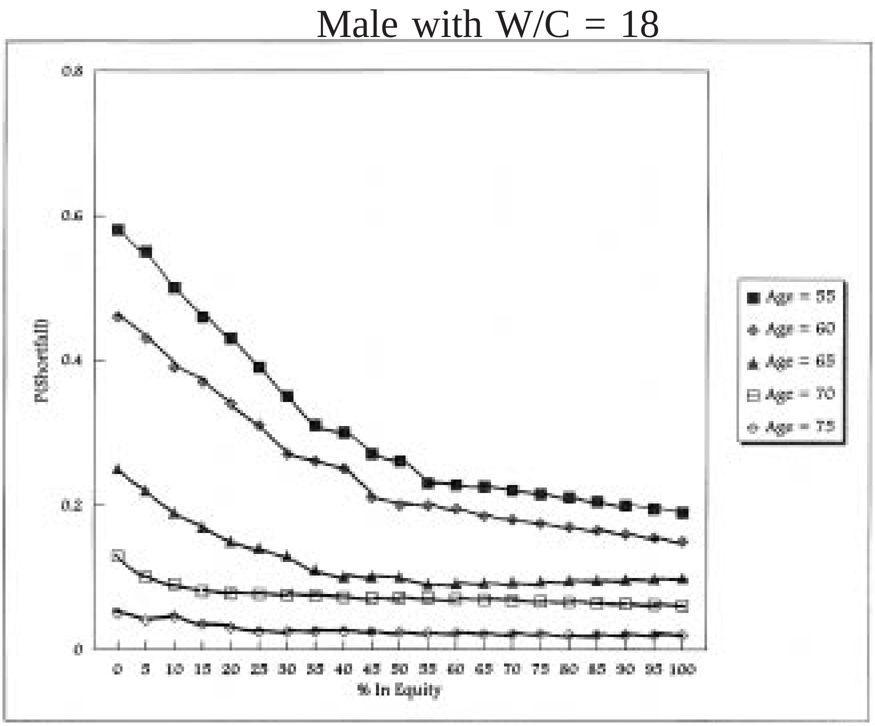

Figure 2. The relationship between the probability of shortfall and the percentage allocation of wealth to equity for males retiring at different ages with a wealth to consumption ratio of 18.

Finally, a woman and a man of the same age and wealth will have different short-fall risks. The woman has a higher risk because of the much longer life expectancy that she has to finance from her retirement savings.[12] Additional equity investment does not allow her to reduce this risk to the same level as for a man, although she should invest more in equity than a man of the same age in most of the simulations.

Table 2 summarizes the results of simulations for a range of values, covering ages 55–75 and *W/C* values from 10 to 27. The upper left and lower right corners are left blank, since they provide no new information. The upper left is all equity for men and women, with very high shortfall probabilities. The lower right represents those fortunate few individuals who have so much relative to their needs that the allocation isn't relevant.

We noted earlier that the use of simulation creates some unevenness, and the function is very flat around the optimum. Table 2 reflects this by displaying ranges of equity allocations, with the average shortfall percentage of the range underneath. The ranges were chosen so that they cover less than two percentage points of shortfall probabilities. We do not think the variation inherent in such a simulation justifies reporting results more precisely.

Table 2. **Summary of Optimal Asset Allocations.** W/C is the wealth-consumption ratio in constant $. The first (top) number in each cell is the optimal percentage (or range of percentages) of equity. The second (bottom) number is the probability of shortfall if this optimal percentage is chosen. For example (in boldface), a female aged 65 with a wealth-consumption ratio of 20 should invest 45–80% of her liquid wealth in equity, to have an 8% chance of shortfall in her desired consumption over her remaining life if she does so. Equity investments outside that range increase the probability of shortfall.

W/C	Age and Sex of Retiree									
	55		60		65		70		75	
	M	F	M	F	M	F	M	F	M	F
10					100	100	90–100	100	70–100	100
					.43	.58	.31	.46	.21	.31
12					100	95–100	80–100	100	65–100	80–95
					.28	.41	.20	.30	.11	.20
14	100	100	85–100	95–100	100	95–100	75–80	90–95	50–100	70–80
	.32	.44	.27	.36	.18	.28	.11	.19	.06	.10
16	100	100	80–100	85–95	60–100	95–100	35–100	60–100	25–100	40–90
	.24	.32	.17	.26	.12	.19	.07	.12	.03	.06
18	80–100	90–95	65–100	80–100	60–70	65–100	30–75	55–60	20–70	20–75
	.17	.23	.12	.18	.06	.13	.03	.06	.01	.03
20	55–80	85–95	50–80	55–80	60	**45–80**	25–75	20–60	20–60	15–75
	.11	.17	.07	.12	.03	**.08**	.01	.03	.004	.01
23	50–60	75–85	30–75	35–75	20–60	20–65	5–75	10–65		
	.06	.09	.03	.06	.01	.03	.004	.01		
27	20–40	30–55	10–75	20–40						
	.01	.03	.01	.01						

2.2 Changing the rates of return

Could the results be simply due to an accidental choice of rates of return? We use historical averages as our choice to represent long-run expectations, but different values are also reasonable. Different asset classes or different estimates of mean and standard deviation may lead to different conclusions.

We tested this possibility by using Canadian long bonds and small cap equity as alternative classes. The long bonds have a mean real return of 2.68% and a standard deviation of 10.61% for 1950–93 [Ho and Robinson 1996], Appendix D. Small cap equity has a mean real return of 11.53% and standard deviation of 16.45% [Ross et al. 1993]. Table 3 presents the results for a female aged 65 with $W/C = 20$. The middle column, Treasury Bills vs. All Equity, repeats the result from Table 1 to provide a benchmark. All Equity simply means the TSE 300. The results are consistent with our previous claim that a substantial allocation to equity reduces the risk of shortfall. The shortfall risk for all combinations of T-bills and bonds is much higher. The risk of shortfall drops well below the benchmark case when combinations of all-equity and small cap equity are tried. Substituting bonds for T-bills, the right-hand column, produces results virtually identical to the benchmark portfolio of T-bills and equity. Similar results obtain for other age and *W/C* combinations, and for males and females.

Table 3. **Probability of Shortfall for Different Risk-Return Combinations.** Each entry is the probability of shortfall for the % invested in the asset named second, which is the more volatile asset. That is, columns 2–4 are the combinations of treasury bills with riskier assets, and column five is the combination of bonds with all equity.

Female aged 65 with $W/C = 20$				
% in the second-named asset	Treasury Bills vs. Bonds	Treasury Bills vs. All Equity	Treasury Bills vs. Small Cap Equity	Bonds vs All Equity
0	.292	.292	.292	.262
5	.271	.240	.215	.215
10	.259	.202	.158	.182
15	.256	.166	.106	.166
20	.239	.132	.068	.142
25	.249	.112	.058	.112
30	***.228***	.092	.051	.098
35	.247	.093	.047	.095
40	.237	.087	.046	.087
45	.230	.070	***.040***	.075
50	.242	.078	***.040***	.078
55	.229	.081	.043	.074
60	.246	.078	.049	.072
65	.245	***.069***	***.040***	***.068***
70	.240	.072	.042	.069
75	.246	.082	.047	.074
80	.244	.078	.050	.071
85	.246	.088	.048	.081
90	.236	.079	.050	.075
95	.242	.090	.059	.088
100	.263	.089	.054	.090

The result that bonds and T-bills have the same effect on the shortfall risk is due to the historical distribution of returns. Bonds have done only a little better than T-bills during our sample period. Booth (1995) argues that the risk premium of Canadian equity over government bonds has already declined and will continue to be lower in the long-run future. We ran the simulation using bonds with arbitrarily higher returns and standard deviations. As expected, the shortfall probabilities are lower for every allocation and the allocation to equity that minimizes shortfall risk is also lower, but a substantial allocation to equity is still advisable in most situations. The results of Table 2 would not be materially altered if bonds with somewhat higher returns were substituted for T-bills.[13]

3. Distribution of bequest

Some retirees might wish to minimize their risk of shortfall, but still retain some reasonable chance of leaving money to their heirs. The probability of starvation can be viewed as a particular statistic of the more general distribution of bequest. Let B denote the random variable which is the amount of wealth that the retiree has at the time of death, in other words the bequest. The probability of starvation **P**(*starve*) can be represented as the probability that the bequest will be less than or equal to zero, $P[B \leq 0]$. The expected bequest is the quantity $E[B]$, the median bequest is $Med[B]$, the standard deviation of bequest is $SD[B]$ and so on and so forth.

In particular, let the random variable $D_n = 1$ when the retiree dies at time period n and $D_n = 0$ at all other time periods n. This discrete binary random variable contains all the necessary information from the mortality table. The random variable will jump from zero to one at the time of death. Also, let W_n denote the wealth of the retiree at time *n provided* that he/she is alive. In the parlance of probability theory, W_n is a conditional random variable, conditional on being alive.

Thus,

$$P[B \leq x] = \sum_{n=0}^{\infty} P[W_n \leq x] P[D_n = 1]$$

is the cumulative density function of the bequest function and can be used to compute assorted quantities.

In Table 4 we present some of the statistics for the bequest function for a female and a male aged 65, with $W/C = 20$ and $C = 1$, and wealth allocated between equity and T-bills. Since we want to know the distribution of the bequest in dollars, we cannot use the ratio W/C, but must express the quantities separately. The bequest is of the same order of magnitude as C; that is, if C is \$10,000, the bequest value shown in Table 4 is multiplied by \$10,000.

Table 4 shows three results. First, the expected bequest and the first and third quartiles all rise monotonically with α, the percentage invested in equity. Second, the female has a lower value of bequest than the male for any quartile at the lower equity allocations, but a higher value for the median and 75th percentile at higher equity allocations. This occurs because the higher rate of return on equity combined with longer life expectancy allows the female to amass more wealth on average, even in retirement.

This distribution would allow a person to trade off a higher bequest for his or her heirs against a higher risk of shortfall. For example, the minimum shortfall probability of 2.7% for the male occurs with 60% equity. Let us assume that $C = \$20{,}000$ and $W = \$400{,}000$. The expected bequest is \$380,000 and the inter-quartile range is \$246,000. He could choose to invest 100% in equity and increase his probability of shortfall to 5.0%, with an expected bequest of \$454,000. Of course, he could also choose to consume more, lowering his W/C ratio, and his shortfall risk-minimizing allocation to equity would increase, as in Table 1.

4. Conclusion and extensions

We have developed a rigorous model to answer the question: how should a retiree allocate investment assets in order to minimize the risk of shortfall in consumption. This model incorporates the mortality tables, gender of the retiree, desired consumption to invested

Table 4. Bequests and Shortfall Probabilities For a person aged 65, with $W = 20$, $C = 1$; therefore $W/C = 20$. The values for the bequest are all in the same magnitude as the initial wealth. For example, look at the entry for a female with the lowest shortfall probability of 6.9% with 65% in equity. If she had wealth of $200,000 and consumption of $10,000, the median value of the bequest is $190,000 in real dollars. The probability of leaving at least $110,000 in real dollars is 75%, and the probability of leaving at least $290,000 is 25%.

	FEMALE				MALE			
	Probability of	Distribution of Bequest			Probability of	Distribution of Bequest		
% in equity	Shortfall	25%	Median	75%	Shortfall	25%	Median	75%
0	.292	0	4.8	10.7	.138	3.7	9.3	14.1
5	.240	.3	6.1	11.7	.099	4.7	10.2	14.9
10	.202	1.4	7.3	12.6	.087	5.8	11.0	15.5
15	.166	2.8	8.5	13.6	.058	7.3	12.0	16.1
20	.132	3.9	9.4	14.8	.055	7.7	12.7	16.8
25	.112	5.1	10.9	15.9	.045	8.8	13.6	17.4
30	.092	6.1	12.3	17.2	.048	9.7	14.7	18.4
35	.093	6.6	13.0	18.3	.035	10.3	15.6	19.6
40	.087	7.7	14.1	19.8	.043	10.8	16.4	20.3
45	.070	8.9	15.5	21.2	.037	11.8	17.0	21.1
50	.078	9.1	16.0	22.7	.038	12.1	18.0	22.6
55	.081	9.2	16.9	23.6	.035	12.3	18.2	23.7
60	.078	10.5	18.3	26.9	**.027**	13.1	19.1	25.4
65	**.069**	11.1	19.1	29.0	.040	13.2	19.0	26.5
70	.072	11.1	19.6	31.9	.042	13.1	19.7	27.8
75	.082	11.4	20.6	34.4	.040	13.7	20.5	29.9
80	.078	11.0	20.8	37.4	.041	13.5	20.8	32.1
85	.088	11.7	21.4	39.0	.047	14.0	21.5	34.4
90	.079	11.6	22.9	42.9	.044	13.8	22.0	35.4
95	.090	11.4	23.1	46.8	.050	14.0	23.1	38.4
100	.089	12.0	24.0	47.8	.055	13.8	22.7	39.0

wealth ratio and rates of return and standard deviation. Unlike other models, consumption is endogenized, because a person can choose to change the risk of shortfall by changing the amount consumed from wealth each year.

The asset allocation literature has ignored these factors, but we find that they are very significant, using a simulation with reasonable values. The principal findings are:

- Retirees should consider their desired consumption, existing wealth, age and gender, before deciding how to allocate their investment assets.
- Retirees in most cases should invest a higher proportion in high risk, high return assets than most planners have traditionally recommended;
- Women need to invest even higher proportions of their wealth in riskier assets than men, because they live longer, on average, and need to earn more from their retirement funds than men do.
- We provide the function for the distribution of the bequest, which would allow a person to trade off higher risk of shortfall against higher expected bequest for the heirs.

We think the results are quite robust, but the model could be improved in several ways, if they prove feasible:

1. Incorporate the joint probabilities of death for a couple, with declining consumption for the survivor compared with the couple. The model as it stands would continue to be valid for single persons.
2. Solve the dynamic problem of reallocation of assets over time. That is, allow the retiree to make a decision now contingent on optimal reallocation at each future date. Our model assumes the allocation is fixed once it is made.
3. Allow a portfolio of more than two assets. This addition may not bring much insight to compensate for the complication.
4. Incorporate indexed and unindexed annuities into the model, including a risk-free annuity. These additions would allow analysis of most of the retirement income sources, since they would include insurance annuities and pension plan income.

In the time-honoured tradition, we leave these complications for future work.

Notes

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2. See, for example, [Benari 1990]; [Butler and Domian 1991]; [Butler and Domian 1993]; [Grauer and Hakansson 1982]; [Lloyd and Modani 1983].
3. We focus on the entire distribution of the life table. Life expectancy is not enough, because the 'force of mortality', as the actuaries refer to it, is not constant over time. If life horizons were exponentially distributed, for example, then the only parameter of importance would be the median life span. The exponential distribution can be fully described by one parameter, its expected value. However, there are many distributions that can have the exact same expected value (or median), mortality patterns among them, and thus life expectancy is not a sufficient statistic for our purposes. The most common functional form used to describe mortality tables is the two-parameter Weibull distribution. Consequently, we categorize our retirees by sex and age. A 76 year old female and a 70 year old male both have the same life expectancy, 12.5 years. However, the 'shape' of their mortality function is different and thus their optimal asset allocation will differ.
4. One can derive the appropriate interest rate that will ensure a perpetuity, without having to resort to equation (5).
5. In reality, a *dynamic* policy would be optimal and may indeed reduce the probability of outliving wealth; however this is beyond the scope of this paper and perhaps may be the subject of future research. A stochastic control theory approach that would incorporate the Hamilton-Jacobi-Bellman equations would be the obvious technique to use.
6. The authors would like to avoid a lengthy discussion about the appropriate model for financial asset returns. Suffice it to say that the Geometric Brownian Motion assumption, which translates into the Lognormal distribution assumption, is still used extensively throughout the continuous time finance literature. In addition, the methodology developed in this paper to analyze the probability of outliving wealth can be

applied to any stochastic specification of returns by employing a Monte Carlo analysis as we do. Thus, for example, if one is convinced that the appropriate model for investment returns is a *Contaminated GBM*, *Jump Poisson Process* or a *GARCH process*, then $\mathbf{R^k}(\vec{\alpha})_i$ would represent a generalized one period return. However, regardless of the exact specification of $\mathbf{R^k}(\vec{\alpha})_i$, it still is an implicit function of the asset allocation proportions $\vec{\alpha}$, and hence can be simulated as such.

7. We further must assume independence between P_t^x and $P[N^* = i]$ for the expression in equation [13] to be valid. Qualitatively this means that mortality must be independent of wealth at all points in time. Thus, a stock market crash will not be allowed to cause heart failures, likewise a sustained bull market does not improve one's health nor does it prolong one's life.
8. We use the Toronto Stock Exchange (TSE) 300 Index, which is a well-diversified benchmark portfolio.
9. In Canada, most of the retirement savings are in special accounts which permit tax deferral, but all withdrawals from these accounts are taxed equally, regardless of how earned. Equity returns on savings outside these accounts are taxed more favourably than bond interest; so our findings in favour of equity are not affected.
10. As a general benchmark, we note that the average family income in the province of Ontario was $57,727 in 1994. The three cases each consider retirement income or withdrawals well below that level.
11. [Ho, Milevsky and Robinson 1994] provide a solution in a simpler model. In practice, the lending rate to individuals will be high enough and the amount loaned constrained so that there will be little reduction in shortfall risk.
12. The difference is 6–7 years in Canada, and is similar in most industrialized nations.
13. We do not include these results in the paper, but they are available from the authors on request.

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