# ESS5203.03; Turbulence and Diffusion in the Atmospheric Boundary-Layer - W2020 PROBLEM SET \#4 - Notes and solutions 

1) Establish formulae for the maximum ground level concentration downwind of a continuous point source at an elevation H above the ground, and the location of the maximum. [You can assume the basic Gaussian Plume formula, constant $U$ etc.]

As in the notes, http://www.yorku.ca/pat/ESS5203/PTDiffusionNotes.pdf the equation for concentration downwind of a source at height H can be written as,

$$
C(x, y, z)=\frac{\mathbf{Q}}{\mathbf{U}} \frac{1}{2 \pi \sigma_{y} \sigma_{z}} e^{\left(\frac{-y^{2}}{2 \sigma_{y}^{2}}\right)}\left[e^{\left(\frac{-(z-H)^{2}}{2 \sigma_{z}^{2}}\right)}+e^{\left(\frac{-(z+H)^{2}}{2 \sigma_{z}^{2}}\right)}\right]
$$

This is assuming no absorption at the ground surface, so $\partial \mathrm{C} / \partial \mathrm{z}=0$.

Symmetry will tell us that the maximum will lie on $y=0$ and $z=0$, but we need o know how $\sigma_{y}$ and $\sigma_{z}$ vary with $x$ in order to establish the maximum value and its location.

In the notes we included the Briggs (1971) formulae, but there are simpler choices with $\sigma_{y}$ and $\sigma_{z}$ proportional to $x$. Bottom line is that we need to find the location and value of the maximum of,

$$
\begin{equation*}
\mathrm{C}(\mathrm{x}, 0,0)=\left(\mathrm{Q} /(\pi \mathrm{U}) \exp \left(-\mathrm{H}^{2} /\left(2 \sigma_{z}^{2}\right)\right) /\left(\sigma_{y} \sigma_{z}\right)\right. \tag{1}
\end{equation*}
$$

With $\sigma_{y}=a x$ and $\sigma_{z}=b x$ we can look for the point where $d C / d x=0$, but for illustration we can also plot $C(x, 0,0)$, and perhaps normalize by $H$, and let $x^{\prime}=x / H, \sigma_{z}{ }^{\prime}=\sigma_{z} / H$ etc.
So $\quad C^{\prime}=\left(\pi U /\left(\mathrm{QH}^{2}\right) C\left(x^{\prime}, 0,0\right)=\exp \left(-1 /\left(2\left(b x^{\prime}\right)^{2}\right)\right) /\left(a b x^{\prime 2}\right)\right.$

As a step in the process one could quickly plot this function in matlab or other software.
$\% P S 4 Q 1$
$a=0.2 ; b=0.2$;
figure
$\mathrm{C}=@(\mathrm{x}) \exp \left(-1 /\left(2^{\star} \mathrm{b}^{\wedge} 2^{\star} \mathrm{x}^{\wedge} 2\right)\right) /\left(\mathrm{a}^{\star} \mathrm{b}^{\star} \mathrm{x}^{\wedge} 2\right)$;
fplot(C, [0.001 20]);
xlabel('x prime'); ylabel('C prime');


So note that there are problems at $x=0$, would need L'Hopitale's rule to demonstrate $C=0$, but the function looks as we might expect and has a maximum. Location and magnitude will depend on $a$ and $b$ values.

To find the location differentiate (2) wrt $x^{\prime}$ and seek location with $d C^{\prime} / d x^{\prime}=0$.

$$
d C^{\prime} / d x^{\prime}=\left[\left(a b x^{\prime 2}\right)\left(1 /\left(b^{2} x^{\prime 3}\right)-2 a b x^{\prime}\right] E\right) /\left(a b x^{\prime 2}\right)^{2}, \quad \text { where } E=\exp \left(-1 /\left(2 b^{2} x^{\prime 2}\right)\right)
$$

This well $=0$ when

$$
\left[\left(a b x^{\prime 2}\right)\left(1 /\left(b^{2} x^{\prime 3}\right)-2 a b x^{\prime}\right]=0 \text {, so }(a / b) x^{\prime-1}=2 a b x^{\prime} \text { and } x^{\prime}=1 /(b \sqrt{ } 2) .\right.
$$

As a check, with $b=0.2$, this is at $x^{\prime}=3.53$.
Also note that the max value of $C^{\prime}$ will be
$\exp \left(-1 /\left(2\left(b x^{\prime}\right)^{2}\right)\right) /\left(a b x^{\prime 2}\right)=(2 b / a) \exp (-1)$
This $=0.736$ with our $a$, $b$ values ( $0.2,0.2$ ), this matches our plot perfectly. Changing $b$ to 0.1 has $x^{\prime}=7.07, C^{\prime}=0.368$, picking numbers off a new plot, as the equations above imply.
2) An accident in a factory near Hwy 7, due N of Petrie between Keele and Jane has led to a continuous, ground level point source release of toxic material at a rate of $0.1 \mathrm{~kg} / \mathrm{sec}$. If the 10 m wind speed is $5 \mathrm{~m} / \mathrm{s}$ from the North and the exposure to a total dose (concentration $x$ time) of $3 \times 10^{-3} \mathrm{kgm}^{-3} \mathrm{~s}$ is dangerous to health, estimate how long we can remain on the campus. Use the Gaussian plume model and sigma curves from my notes or any textbook (but say which since they differ) and determine (and report) the stability class based on conditions at the time you work on the problem.

Writing this on the morning of 11 April, 10am EDT, 1400 UTC. Weather conditions at York have 10 m wind speed $=4.2 \mathrm{~ms}-1, \mathrm{~T}=3 \mathrm{C}$ but note $\mathrm{T}(9.5 \mathrm{~m})-\mathrm{T}(1 \mathrm{~m})=-0.5 \mathrm{C}$. Skies were clear (Downwelling solar radiation about $400 \mathrm{Wm}^{-2}$ ).

The solar radiation, time of day, temperature difference and moderate winds suggest slightly unstable conditions so I would classify this as B-C from slide 10 in my Diffusion notes based on ARL guidance. My inclination would be to assume $C$ to reduce the dispersion and provide a conservative estimate from a health perspective. Also we are using a $5 \mathrm{~m} / \mathrm{s}$ wind in the calculation.

As in Qu 1 one assumes no absorption by the ground so that the Gaussian plume prediction from a source with $\mathrm{H}=0$ is doubled. i.e 2 x the concentrations from the unconstrained continuous point source formula,

$$
\mathbf{C}(\mathbf{x}, \mathrm{y}, \mathrm{z})=\frac{\mathbf{Q}}{\mathbf{U}} \frac{1}{2 \pi \sigma_{\mathrm{y}} \sigma_{\mathrm{z}}} \mathbf{e}^{\left(\frac{-\mathrm{y}^{2}}{2 \sigma_{\mathrm{y}^{2}}^{2}}\right)} \mathrm{e}^{\left(\frac{-\mathbf{z}^{2}}{2 \sigma_{\mathrm{z}}^{2}}\right)}
$$

We are not told exactly where the source is but we are concerned with a position on the campus, so again one should be conservative and look for the worst case scenario. Looking at maps or Google Earth the likely sources near Hwy 7 are about 2.5 km from the Petrie building so I will use that as the distance from the source.

That gives me $\sigma_{y}$ and $\sigma_{z}$ values of 300 m and 150 m reading off the $\mathrm{P}-\mathrm{G}$ curves for stability class $C$. If we are directly down wind and at ground level, $y=z=0$, and we then have, multiplying (1) by the factor 2 ,

$$
C=Q /\left(U \sigma_{y} \sigma_{z}\right)=0.1 /\left(5^{*} \Pi^{*} 300^{*} 150\right)=1.415 \times 10^{-7} \mathrm{kgm}^{-3}
$$

The concern is dosage ( $\mathrm{D}=3 \times 10^{-3} \mathrm{kgm}^{-3} \mathrm{~s}$ is dangerous to health) and so the time we could remain $\mathrm{T}=\mathrm{D} / \mathrm{C}$ is $21201 \mathrm{~s}=5.8895$ hours. Better to not push that limit!
3) Read up on the ratio between Lagrangian and Eulerian integral time scales and explain why they are different. With the data in Assignment3_Data.txt", after rotation into a coordinate system with $\mathrm{V}=\mathrm{W}=0$ (Qu2 of Assignment 3), compute and plot the Eulerian autocorrelations $R_{u}(\xi)$ etc for $u^{\prime}, v^{\prime}$ and $w^{\prime}$ and, if you can, determine the integral time scales. (the integral may not converge numerically so estimate in that case).

See notes for definitions of Eulerian vs. Lagrangian. The Eulerian time scale is typically smaller. When moving with an air parcel (as in Lagrangian motion), the "memory" of the flow is longer as the same (or similar) eddies are moving with the parcel.

See graphs on the next page for calculations of the Eulerian autocorrelations. If we assume the decay is exponential, we can estimate the integral as the value of $\rho$ (or R following the terminology in the question) at a value of 1/e. This gives length-scales of $27 \mathrm{~s}, 52 \mathrm{~s}$, and 7.2 s for $\mathrm{u}, \mathrm{v}$, and w respectively. Alternately we can calculate a best-fit exponential of $R=\exp \left(-t / T_{E}\right)$, which gives $T_{E}$ of 42s, 48s, and 7.7 s .

Strict integration causes problems since there are negative values at larger $t$. So, a cut-off must be determined.

Note that Ru doesn't go to zero as expected which usually indicates a trend in the time series (i.e. wind speed increasing over the 30 min period).




