Physics 4070/5090: Stars and Nebulae: W11-12: Prof. Hall
HW#1 Answers. Due START OF CLASS Thu. Jan 12. Total marks: 6.5 (8)

For Partial Credit, Always Show Your Work!
Proper Use Of Significant Figures Is Required!
e.g. $(7 \times 10^3) \times (6.0001 \times 10^4) = 4 \times 10^7$ **or** 4.2×10^7

1. A. [2 marks] The general formula for μ, the average mass per free particle in units of the hydrogen mass $m_H = 1.008$ u (atomic mass units), is $\mu = \sum_i m_i n_i / \sum_i m_H n_i$, where each i represents a different type of particle and the n_i can be true or relative particle densities.

Consider a gas of pure He ($m_{He} = 4.003$ u) which is partially ionized to He^{+1} so that it consists of 3 species of free particles: neutral helium, singly-ionized helium, and free electrons. Define x to be the He^{+1} ionization fraction (the fraction of all helium nuclei that are orbited by one electron instead of two). What is the formula for μ in terms of x, m_{He} and m_H?

B. [0.5 mark] Check your answer: for $x = 0.3186$, you should obtain $\mu = 3.012$, and you should obtain sensible values in the limits $x = 0$ and $x = 1$.

ANSWER. Define the neutral helium number density as $n_0 = (1-x)\rho_{tot}/m_{He}$ the singly-ionized helium number density as $n_+ = x\rho_{tot}/m_{He}$, and note that the free electron density is $n_e = n_+$. Then:

$$\mu = \frac{n_0 m_{He} + n_+(m_{He} - m_e) + n_e m_e}{m_H[n_0 + n_+ + n_e]} = \frac{n_0 m_{He} + n_+(m_{He} - m_e + m_e)}{m_H[n_0 + 2n_+]}$$

$$\mu = \frac{m_{He}}{m_H} \frac{(1-x)\rho_{tot}/m_{He} + x\rho_{tot}/m_{He}}{[(1-x)\rho_{tot}/m_{He} + 2x\rho_{tot}/m_{He}]} = \frac{m_{He}}{m_H} \frac{\rho_{tot}}{[(1+x)\rho_{tot}]} = \frac{m_{He}}{m_H} \frac{1}{1+x}$$

$$\mu = \frac{m_{He}}{m_H} \frac{1}{1+x}$$

$$\mu(x = 0.3186) = \frac{4.003}{1.008} \times \frac{1}{1 + 0.3186} = 3.012$$

For $x = 0$, we have pure He^0 with $\mu(x = 0) = 4.003/1.008 = 3.971$ as expected. For $x = 1$, half the particles are He^{+1} and half electrons, with $\mu(x = 1) = 1.986$ as expected (half that of $x = 0$).

C. [1.5 marks] **FOR GRAD STUDENTS; extra credit for undergrads.** What is the simplest expression for the even more general formula for μ when each element i of mass m_i and number density n_i has j ionization states [including the neutral state $j = 0$], each with ionization fraction x_{ij} ($\sum_j x_{ij} = 1$)? [Hint: how many electrons per ion?] Check your answer for a gas of 50% H and 50% He heated to the point where the hydrogen has $x_{10} = 1/6$ and $x_{11} = 5/6$ and the helium is 1/2 He^0, 1/3 He^{+1} and 1/6 He^{+2}. Assuming the ionization fractions are exact, you should find $\mu = 1.420$.

ANSWER. In the formula for μ we must replace n_i with a sum over j, $\sum_j n_{ij}$. (We do *not* need to consider the individual m_{ij} separately; although ion j's mass is reduced by jm_e, those j electrons can still be counted with their parent atoms: $\sum_j n_{ij}m_{ij} = \sum_j(n_{ij}(m_i - jm_e) + jn_{ij}m_e) = \sum_j n_{ij}m_i$.) Furthermore, $n_{ij} = x_{ij}n_i$ and $\sum_j x_{ij} = 1$, so in the numerator the sum over j reduces back to n_i. And each ion in ionization state j produces j electrons, so the number of free particles per ionization state is $1 + j$:

$$\mu = \frac{\sum_i(m_i \sum_j n_{ij})}{m_H \sum_i \sum_j[x_{ij}n_i(1+j)]} = \frac{\sum_i(m_i n_i \sum_j x_{ij})}{m_H \sum_i \sum_j[x_{ij}n_i(1+j)]} = \frac{\sum_i m_i n_i}{m_H \sum_i \sum_j[x_{ij}n_i(1+j)]}$$

$$\mu = \frac{\sum_i m_i n_i}{m_H \sum_i \sum_j[x_{ij}n_i(1+j)]}$$

Check: if n is the original helium number density, then we can write $n_H = n$, $n_0 = n/2$, $n_1 = n/3$ and $n_2 = n/6$. Then

$$\mu = \frac{nm_H + nm_{He}}{m_H[\frac{n}{6}(1+0) + \frac{5n}{6}(1+1)] + m_H[\frac{n}{2}(1+0) + \frac{n}{3}(1+1) + \frac{n}{6}(1+2)]}$$

$$\mu = \frac{m_H + m_{He}}{m_H(\frac{1}{6} + \frac{10}{6} + \frac{3}{6} + \frac{4}{6} + \frac{3}{6})} = \frac{6}{21}\left(1 + \frac{m_{He}}{m_H}\right) = \frac{2}{7}(1 + 3.971) = 1.420$$

2. [4 marks total] Let's consider whether it is reasonable for models of the Sun's structure that desire 1% accuracy in their density and pressure profiles to assume $\rho = 0$ and $P = 0$ at the Sun's surface.

A) [1 mark] The pressure at the Sun's photosphere is inferred from observations to be $P = 10^5$ dyne/cm^2. Assuming that the ideal gas law applies, with $\mu = 0.5$ (appropriate for completely ionized pure hydrogen), what is the density at the Sun's photosphere? By what multiplicative factor does it differ from Earth's sea-level atmospheric density of 1.2×10^{-3} g cm^{-3}?

ANSWER: $\rho = P\mu_M/R_gT$ dyne cm^{-2} g / (dyne cm) = **1.04** $\times 10^{-7}$ **g/cm**$^{-3}$ **That density is 1.2**$\times$**10**^{4} **smaller than Earth's sea-level atmospheric density.**

B) [2 marks] Assuming that the ideal gas law applies throughout the interior of the Sun, with $\mu = 0.5$ appropriate for completely ionized pure hydrogen, what is the minimum average pressure inside the Sun (below the photosphere)? (To find that, you'll need the average mass density inside the Sun — what is that density?)

ANSWER: The average mass density inside the Sun is just$3M_\odot/4\pi R_\odot^3$**, where** $M_\odot = 2.0 \times 10^{33}$ **g and** $R_\odot = 7.0 \times 10^{10}$ **cm, so** $\bar{\rho} =$ **1.4 g cm**$^{-3}$**. Now, because** $P = \rho R_g T/\mu_H$ **applies throughout the Sun, and the Sun must be at least as hot below the photosphere as it is in the photosphere, the minimum average pressure inside the Sun is**
$P = \bar{\rho} R_g T_{\text{eff}}/\mu_H = 1.4 \times 8 \times 10^7 \times 5800/0.5 = 1.3 \times 10^{12}$ dynes cm^{-2}

C) [1 mark] Given the results of part A and B, what percentage of the Sun's average density is the Sun's photospheric density? And what percentage of the Sun's minimum average pressure is the Sun's photospheric pressure? Given those values, is it acceptable for models of the Sun's structure to assume $\rho = 0$ and $P = 0$ at the Sun's surface and still expect 1% accuracy in their density and pressure profiles?

ANSWER: The Sun's photospheric density is just 0.0000079% (a fraction 7.9E-8) of its average density, and its photospheric pressure is only 0.0000077% (a fraction 7.7E-8) of the Sun's minimum average pressure. So yes it's acceptable for models of the Sun's structure to assume $\rho = 0$ **and** $P = 0$ **in the photosphere and still have 1% accuracy in their density and pressure profiles.**