Convection (BV3: 5.1-5.3, but the discussion is sometimes poor)

Imagine a small element of mass is in equilibrium with its surroundings at pressure P, density ρ & temperature T. (Assume that radiation pressure is negligible, so $P = P_{gas}$.) Suppose it is displaced *downwards* by a random perturbation and finds itself surrounded by gas at pressure $P + dP$, density $\rho + d\rho$ & temperature $T + dT$. The element will adjust to some new P', ρ' & T'. We assume that the displacement takes place slowly enough that the element stays in pressure equilibrium with its surroundings ($P' = P + dP$), but quickly enough that the displacement is **adiabatic**: no heat energy is transferred into or out of the mass element (which is *not* the same as saying its temperature is constant). The new density of the element is given by

$$\rho' = \rho + \left(\frac{d\rho}{dP}\right)_{ad} \times dP$$

where the subscript *ad* indicates an adiabatic change.

If the increase in the density of the mass element as the pressure rises is greater than the increase in the density of its surroundings, it will continue to sink. (And a mass element displaced upwards which finds itself still less dense than its surroundings as the pressure eases will continue to rise.) In other words a region will be **unstable to convection** of gas if:

$$\rho' = \rho + \left(\frac{d\rho}{dP}\right)_{ad} \times dP > \rho + d\rho \quad \longrightarrow \quad \left(\frac{d\rho}{dP}\right)_{actual} < \left(\frac{d\rho}{dP}\right)_{ad}$$

It is more common to write this condition in terms of a temperature gradient with pressure, using the ideal gas law $\rho = \frac{\mu_H P}{R_g T}$ assuming constant μ_H:

$$\left.\frac{\mu_H}{R_g}\frac{d(P/T)}{dP}\right|_{actual} = \frac{\mu_H}{R_g}\left(\frac{1}{T} - \frac{P}{T^2}\frac{dT}{dP}\right)_{actual} < \frac{\mu_H}{R_g}\left(\frac{1}{T} - \frac{P}{T^2}\frac{dT}{dP}\right)_{ad} = \left.\frac{\mu_H}{R_g}\frac{d(P/T)}{dP}\right|_{ad}$$

On both sides we factor out μ_H/R_gT, subtract 1, then multiply by -1:

$$1 - \left.\frac{dT/T}{dP/P}\right|_{actual} < 1 - \left.\frac{dT/T}{dP/P}\right|_{ad} \quad \longrightarrow \quad \boxed{\left(\frac{d\ln T}{d\ln P_{gas}}\right)_{actual} > \left(\frac{d\ln T}{d\ln P_{gas}}\right)_{ad}}$$

The above is the Schwarzschild criterion for convective instability, using the logarithmic gradient of temperature with pressure (called ∇ in BV3).

To determine the logarithmic gradient of temperature with pressure under adiabatic conditions, we can differentiate the ideal gas law $P = \frac{\rho k T}{\mu m_H}$:

$$dP = \frac{kT}{\mu m_H} d\rho + \frac{k\rho}{\mu m_H} dT = \frac{kT\rho}{\mu m_H}\frac{d\rho}{\rho} + \frac{kT\rho}{\mu m_H}\frac{dT}{T} \quad \longrightarrow \quad \frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}$$

where in the last step we used $\frac{kT\rho}{\mu m_H} = P$. To find $d\ln P/d\ln T$, we need to express $d\rho$ above in terms of dT. To do so we use the fact that in an adiabatic process the change in heat energy is $dQ = dE + PdV = 0$, so $dE = -PdV$, where PdV is the work done by the system and dE is the change in the internal energy of the system. The internal energy for a single particle is written $C_V T$, where C_V is the specific heat at constant volume, so the internal energy of N particles is $E = NC_V T$. Thus we have $NC_V dT = -PdV$ for an adiabatic process, and we can relate dV to $d\rho$ using the definition $\rho = N\mu m_H/V$:

$$d\rho = -\frac{N\mu m_H}{V^2} dV = -\frac{\rho}{V} \times -\frac{NC_V dT}{P} = \frac{\rho^2 C_V}{\mu m_H P} dT$$

Plugging back into the differentiated ideal gas law:

$$\frac{dP}{P} = \frac{kT}{kT}\frac{\rho}{\mu m_H}\frac{C_V}{P} dT + \frac{dT}{T} = \frac{PC_V}{kTP} dT + \frac{dT}{T} = \frac{dT}{T}\left(1 + \frac{C_V}{k}\right) = \frac{dT}{T}\left(\frac{k + C_V}{k}\right)$$

Using the definition of the specific heat at constant pressure, $C_P = C_V + k$:

$$\frac{dP}{P} = \frac{dT}{T}\left(\frac{C_P}{C_P - C_V}\right) = \frac{dT}{T}\left(\frac{C_P/C_V}{C_P/C_V - 1}\right) = \frac{dT}{T}\left(\frac{\gamma}{\gamma - 1}\right)$$

where $\gamma = C_P/C_V$ is the ratio of specific heats. So finally we have

$$\left(\frac{d\ln T}{d\ln P}\right)_{actual} > \frac{\gamma - 1}{\gamma} = \left(\frac{d\ln T}{d\ln P}\right)_{ad}$$

as the condition for convective instability in an ideal gas.

To determine γ, first note that the internal energy of a single particle is $\frac{1}{2}kT$ per degree of freedom. For monatomic gas, a good approximation deep inside stars, the only degrees of freedom are motions in the 3 space dimensions, so $C_V = \frac{3}{2}k$, $C_P = \frac{5}{2}k$, $\gamma = C_P/C_V = 5/3$, and $\frac{\gamma-1}{\gamma} = 0.4$.

Convection, cont'd. If the gradient of log$T$ with log$P$ in some region of a star is greater than the adiabatic value of $\frac{\gamma-1}{\gamma}=0.4$, energy will cease to be transported outwards solely by radiation in that region. Instead, some gas will be displaced upwards, where it is hotter than its surroundings and therefore buoyant; the gas will continue to rise, carrying energy upwards until the gas has cooled enough through adiabatic expansion (and some heat transfer to its surroundings) to match the temperature of its surroundings. And some gas will be displaced downwards, where it is cooler and therefore less buoyant than its surroundings; it will keep sinking until it has heated sufficiently through compression (and absorbing some heat from its surroundings) to reach the same temperature as its suroundings.

Thus, convection transfers energy outwards, to larger radii: consider that PdV work is done *by* a rising gas bubble, increasing the internal energy of its surroundings at larger radii, and done *on* a falling gas bubble, decreasing the internal energy of its surroundings at smaller radii. Convection therefore transfers energy from smaller to larger radii and reduces the vertical temperature gradient in any region where it operates.

Convection is so efficient at transporting energy that ∇_{actual} saturates at ∇_{ad} in convective regions. If the gradient is slightly below ∇_{ad} and then increases, at ∇_{ad} convection sets in and carries energy so efficiently that the temperature gradient is kept at the adiabatic value. So, for stars with a given stellar central temperature T_c, there is a lower limit to the surface $T_{\rm eff}$ which occurs when the star is fully convective.

To determine when convection occurs, note that $d\ln T(r)/d\ln P(r) = \frac{(dT/dr)/T}{(dP/dr)/P}$. Using our stellar structure expressions for dT/dr and dP/dr,

$$\frac{d\ln T}{d\ln P}=\frac{3\kappa_R L(r)P(r)}{64\pi\sigma_{sb}[T(r)]^4 GM(r)}$$

In practice there are three main ways in which convection occurs:
1) Large κ_R, which traps radiation and increases the temperature gradient until convection sets in; e.g., at $T_{\rm eff}$=6000 K H^- & excited H increase κ_R;
2) Large L relative to $\sigma_{sb}T^4$ increases dT/dr; e.g., cores of massive stars;
3) Large C_V makes $\gamma = 1+\frac{k}{C_V}\to 1$ and $\frac{\gamma-1}{\gamma}\to 0$, which makes it easier for convection to occur; e.g., when H or He start ionizing or H_2 starts dissociating, gas can absorb lots of energy dE for a small dT (large C_V).

Stellar Evolution (see 'Lecture 10' handout, University of Turku):

• During its formation, a solar-mass star's core contracts and heats up as H fuses to He. At the end of the star's MS lifetime, fusion slowly ceases in the core but continues in a shell around the core, first by the p-p chain and then by the CNO cycle, generating the luminosity needed to resist the pressure of the rest of the star's mass weighing down from above. But that same luminosity used to hold up the star's *entire* mass. Now it has less mass to hold up, so it does work and expands the star's envelope. That doesn't change the pressure in the H-fusing shell, so the luminosity of the star stays constant while it expands, meaning that its surface temperature ________________, and the star moves ________________ in the H-R diagram.

• The surface convection layer gets deeper and deeper. If $T_{\rm eff} < 5000\,$K, it gets deep enough to 'dredge up' material processed by nuclear reactions.

• There is a lower limit to $T_{\rm eff}$ at a given M_S. The mass of a star determines its central conditions, which determine the fusion rate that produces the star's luminosity. As shell H-fusion continues, the increasing-mass He core shrinks, increasing T and thus the fusion rate and the luminosity generated in the shell. If the luminosity generated at small radii creates too large a temperature gradient, convection takes over and maintains a maximum $\frac{dT}{dr}$ (and thus $\frac{dL}{dr}$). As a post-MS star's (fully convective) envelope expands, the opacity decreases until all the star's luminosity escapes. That point is reached at a finite $T_{\rm eff} \simeq 4500\,$K for $M_S = M_\odot$ ($T_{\rm eff} \simeq 2500\,$K for $M_S = 0.08\ M_\odot$). Once the star reaches its minimum $T_{\rm eff}$, its increasing luminosity means that the star must ________________ and move ________________ in the H-R diagram. That sequence is known as the *red giant branch*; motion along it stops once core He fusion starts.

• Shell H-fusion continues after core He-fusion starts (the *helium flash*), but even together the luminosity is reduced, so the star dims, shrinks and heats up a bit (it moves from the giant branch to the *horizontal* branch).

• Eventually, a solar-mass star ($0.4M_\odot < M_S < 3M_\odot$) has a double-shell configuration: a core of inert carbon, surrounded by a shell fusing He to carbon, surrounded by inert He, surrounded by a shell fusing H to He, topped by the star's mostly-H envelope. At this point the star retraces its evolution up the red giant branch, but offset slightly to the left (higher $T_{\rm eff}$); the star is on the *asymptotic giant branch.*

• After the AGB phase comes the PN phase, then the WD phase.

• High-mass stars ($> 3M_\odot$ or so) become red giants too, but do not show such large luminosity increases on the RGB. Instead, shortly after such a star reaches the RGB, He fusion begins and the star makes a *blue loop excursion.* Once He is exhausted in the core, double-shell fusion begins and the star returns to the RGB. If carbon fusion begins, another blue loop excursion occurs, and similarly for other fusion reactions. These blue loop excursions are very short (tens of thousands of years at most).

• Stars of up to $\sim 8M_\odot$ become *white dwarfs* (WDs). After all nuclear burning occurs, what's left in those stars is a degenerate core with $M < M_{ch}$, where M_{ch} is the *Chandrasekhar mass* of $\sim 1.4\ M_\odot$ (M_{ch} can be larger if rotation or hydrogen abundance is significant).

• Stars of $> 8M_\odot$ fuse dfferent elements until iron is produced in their cores. Fe will absorb energy by either fusion or fission, so thermonuclear reactions involving Fe absorb energy in the core. Other reactions also occur, including $p + e^- \to n + \nu_e$. Energy absorption and reduced electron pressure send the core into near free-fall, which ends when nuclear densities are reached and a *neutron star* is formed (or when a *black hole* is formed, for stars of $\sim 20M_\odot$ or more).

• If a neutron star is formed ($8M_\odot < M_S < 20M_\odot$), the shells outside the core impact the incompressible neutron star and bounce off, leading to a (Type II) *supernova* explosion (at least in theory; generating such an explosion seems to require a boost from the neutrinos produced with the NS forms, and from asymmetries present only when fully three-dimensional simulations are run). Left behind is a NS of mass no more than 3 or 4 $M_\odot$ or so; the exact upper limit is uncertain.

• Rotating accretion onto a black hole formed in a rotating star may be able to cause a *gamma-ray burst*, possibly accompanied by a *hypernova*, but this theory is a work in progress (for example, there are at least two types of GRBs, and one type is probably caused by NS-NS or NS-BH mergers).

• High-mass stellar evolution (including that of Population II and III stars) is still a topic of current research! For example, Type Ia SNe were thought to occur in old stellar populations, but in the past few years have been found by the CFHT Supernova Legacy Survey to be associated with both old and young (high-mass) stellar populations...