(1) Let $f(z) \in \mathbb{Z}[z]$ be a polynomial of degree $d \geq 2$ with $f'(0) = 0$.
   (a) Show that if $p \mid f^n(0)$ and $p \mid f^m(0)$, then $p \mid f^\gcd(n,m)(0)$. In particular, if $p \mid f^r(0)$ and $r$ is the smallest such positive integer, then $p \mid f^n(0)$ if and only if $r \mid n$.
   (b) Show that if $p^a \| f^r(0)$, then $p^a \| f^{r+j}(0)$ for all integers $j \geq 1$. Here $p^a \| N$ means $p^a \mid N$, but $p^{a+1} \nmid N$.
   (c) Show that if every prime dividing $f^n(0)$ also divides $f^m(0)$, for some $m < n$, then
   \[
   \log |f^n(0)| \leq d^{1+n/2} \log f(0) + O(n \log n).
   \]
   (d) Assuming that 0 is not preperiodic, show that for all but finitely many $n$ there exists a prime $p_n \mid f^n(0)$ such that $p_n \nmid f^m(0)$ for all $1 \leq m < n$ (this is a primitive prime divisor of the $n$th term in the orbit).
(2) (From Silverman’s book) Let $f(z) \in K(z)$ be a rational function of degree $d \geq 2$, and assume that $P \in \mathbb{P}^1$ is a point with $e_P(f^n) \geq C^n$ for some $C > 1$. Prove that $P$ is preperiodic, and that the periodic cycle it lands in has period at most $(2d - 2)/(C - 1)$.
(3) In class we proved some theorems based on the hypothesis that $f^n$ is not a polynomial for any $n$. Prove that if $f$ is a rational function and $f^n$ is a polynomial, for some $n$, then already $f^2$ is a polynomial. Further prove that if $f^2$ is a polynomial, then either $f$ is a polynomial or $f$ is affine conjugate to $f(z) = 1/z^d$.