LIFE INSURANCE AND PENSIONS

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Introduction

Life insurance is the business of insuring human life: in return for a premium payable in one sum or installments, an insurance company guarantees to pay an amount of money on the death of the insured person. Life insurance comes in many forms.

A typical *whole life policy* may stipulate that the insured pay, say, $500 each year as long as she is alive, and that the insurance company pay her beneficiary $50,000 when she dies.

The mutual obligations of a *term policy* hold for a specified period of time. For example, under a three-year term policy for $50,000, an insured may be required to pay $350 each year for three years, provided he is alive when the premium is due. If he dies within the three-year period, his beneficiary receives $50,000. At the end of the three-year period, the insurance contract expires.

An *endowment policy* is similar to a term policy but the policy amount is paid even if the insured survives the term of the policy.

In contrast, *annuities*—or *pensions*, as they are better known—are in a sense the opposite of life insurance. In life insurance, the company pays when the insured dies; under a pension, the company pays for as long as the insured is alive. As with insurance, annuities come in many forms.

A common type is the *life pension*, under which the insured pays a certain amount in advance and the insurance company makes a specified monthly or annual payment until the death of the insured.

Insurance companies sell both life insurance and annuities. In both cases the company assumes a risk, the uncertainty concerning the time of death of an individual. In a whole life policy, for example, the policy amount will be paid eventually, but the total premiums collected will vary depending on how long the insured remains alive. In a life pension, uncertain is the length of the period during which the individual will live to receive the pension.

Although it is not possible to forecast with certainty when an individual will die, the mortality experience of a large number of individuals may be stable enough to be predictable with a fair degree of accuracy. It is upon this experience that insurance companies draw to determine the premiums appropriate for a given type of policy and policy amount.

Mortality tables

The mortality pattern for a nation or other large group of individuals is usually summarized in a mortality table. A portion of such a table is shown in Table 1.

Table 1
Portion of mortality table, males

Age (1)	Death rate* (2)	Deaths in year (3)	Number living (4)
0	1.48	1,480	100,000
1	0.10	98	98,520
2	0.08	79	98,422
...	...	...	...
60	2.00	1,558	77,861
61	2.18	1,667	76,303
62	2.38	1,777	74,636
63	2.59	1,885	72,859
64	2.80	1,990	70,974
...	...	...	...
109	100.00	1	1

*Annual, multiplied by 100

The starting point is the assumption that death rates at each age are reasonably stable over time, and are thus predictable with sufficient accuracy to form a basis of planning by insurance companies. Estimates of these death rates are shown in Column (2) of Table 1. We see, for example, that 2.18% of males aged 61 are expected to die before they reach age 62; 2.38% of males aged 62 are expected to die before reaching age 63; and so on.

Columns (3) and (4) show that out of 100,000 males born, 1.48% or 1480 are expected to die before they reach age 1. Therefore, it is expected that there will be 98,520 survivors at the end of the year. Of these, 0.10% or 98 are expected to die before reaching age 2, leaving 98,422 survivors of age 2. And so on, until by age 110 there are no survivors left of the original group of 100,000.

It should be noted that death rates vary not only with age, but also with gender (the mortality rates for women are generally lower than those for men), occupation, social or economic class, country or region, racial origin, and other factors. Insurance companies have found that the mortality rates for people who purchase life insurance are different from, and generally lower than, the rates for the population as a whole. Also, the mortality rates of people who purchase life insurance are generally higher than those of people who purchase annuities. As a result, there exist a number of mortality tables, each based on the experience of particular groups of people and used for particular purposes.

Determining life insurance premiums

To illustrate how insurance companies set the premiums for particular types of policies and policy amounts, let us consider the case of a five-year term policy for $10,000 issued to a 60-year-old man. The insured undertakes to pay a premium of $\$P$ at the beginning of each policy year for five years (the first premium payable immediately), provided that he is still alive. In return, the insurance company undertakes to pay the insured's beneficiary $10,000 in the event that the insured dies within the next five years. The mutual obligations end exactly five years from the date of issue of the policy.

From the point of view of the insurance company, the possible events and their consequences are as shown in Figure 1.

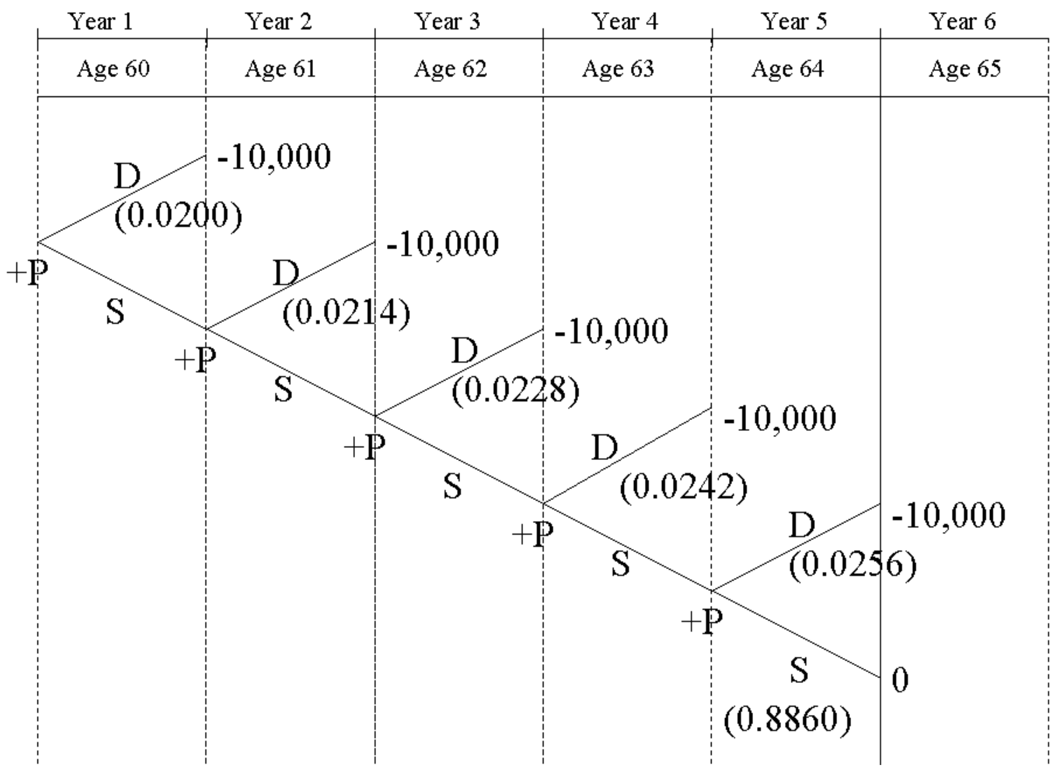

Figure 1
Term insurance

In Figure 1, revenue is indicated by a positive, and outlay by a negative sign. For simplicity we assume that, if the insured dies during a policy year, the payment to the beneficiary is made at the end of that year. The insured pays the first premium P at the beginning of the first policy year. During the first year, one of two things will happen: either the insured dies (D) within the year, in which case \$10,000 is paid out at the end of the year, or he survives (S) the year, in which case he pays the second premium of P at the beginning of the second policy year. In the second policy year, again one of two things will happen: either he dies, and a payment of \$10,000 is made, or he survives and pays a premium of P at the beginning of the third policy year. And so on, to the end of the term.

Overall, then, one of six events will happen: the insured dies at age 60 (i.e., during the first policy year), or at age 61, or 62, or 63, or 64, or he survives age 64. From Table 1, we note that of 77,861 men of age 60, 1,558 or 2.00% are expected to die at age 60; 1,667 or 2.14% at age 61; 1,777 or 2.28% at age 62; 1,885 or 2.42% at age 63; and 1,990 or 2.56% at age 64, leaving 68,984 or 88.60% survivors of age 65. Thus, the probability that a 60-year old man will die at age 60 is 0.0200; at age 61, 0.0214; and so on. The probability that a 60-year old man will survive age 64 is 0.8860. Counting net income (total premiums minus outlays), and ignoring for the moment that payments are made at different points in time, we note that if the client dies at age 60, the company's net income is $(P-10,000)$; if he dies at age 61, the net income is $(2P-10,000)$; ... and if the client survives age 64, the net income is $5P$. In other words, net income can be regarded as a random variable, a function of another random variable—the age at death. The probability distribution of net income is shown in Columns (2) and (3) of Table 2. (An alternative description of the reasoning can be found in the presentation file `PensIns.pps`.)

Table 2
Probability distribution of net income,
five-year term policy for \$10,000 at age 60

Age at death (1)	Probability (2)	Net income (3)	Discounted net income (4)
60	0.0200	$P-10000$	$P-10000v$
61	0.0214	$2P-10000$	$P+Pv-10000v^2$
62	0.0228	$3P-10000$	$P+Pv+Pv^2-10000v^3$
63	0.0242	$4P-10000$	$P+Pv+Pv^2+Pv^3-10000v^4$
64	0.0256	$5P-10000$	$P+Pv+Pv^2+Pv^3+Pv^4-10000v^5$
65+	0.8860	$5P$	$P+Pv+Pv^2+Pv^3+Pv^4$
	1.0000		

Net income, as measured above, does not take into account the different value of money at different points in time and is an inappropriate measure of the outcomes of this policy for the insurance company. For if the company can invest its funds at an interest rate i, then \$1 invested now will be worth $\$(l+i)$ one year from now,

$\$(1+i)^2$ two years from now, and $\$(l+i)^n$ n years from now. Likewise, \$1 payable one year from now is worth $\$v$ now, where $v = 1/(l+i)$; \$1 payable n years from, is now worth $\$v^n$. The present value of the various receipts and payments associated with each event is shown in the last column of Table 2. For example, $\$10,000v^2$ is the value now of \$10,000 payable two years from now; $P + Pv + Pv^2$ is the value now of P payable at the beginning of this and each of the next two years.

Col. (4) of Table 2 shows the company's discounted net income for each age at death of the insured. Thus, the discounted net income will be $P - 10,000v$ with probability 0.2000, $P + Pv - 10,000v^2$ with probability 0.0214, and so on.

The question, once more, is: what should P be for given v? The expected average discounted net income of the insurance company is calculated by multiplying the entries in the last column of Table 2 by the corresponding probabilities and summing up the results:

$$\begin{aligned}(P - 10000v)(0.0200) + (P + Pv10000v^2)(0.0214) + \cdots \\ + (P + Pv + Pv^2 + Pv^3 + Pv^4)(0.8860).\end{aligned}$$

After grouping terms, this expression can also be written as

$$\begin{aligned}P(1 + 0.98v + 0.9586v^2 + 0.9358v^3 + 0.9116v^4) \\ - (200v + 214v^2 + 228v^3 + 242v^4 + 256v^5).\end{aligned}$$

If we assume that the insurance company can invest its funds at 5% per annum, then $v = 1/(l + 0.05) = 0.9524$, $v^2 = 0.9070$, etc. Substituting these values in the last expression, we find that the expected discounted net income is

$$(4.3612)P - (981.2123).$$

It does not make sense in this case to ask which value of P maximizes the expected discounted net income.* Obviously, the larger the premium, the lower are the sales of the company. Since we have not taken demand into account, we calculate instead that value of P at which the expected discounted net income equals zero. This is the *break-even*, or, in the terminology of the industry, the *pure premium.*

The pure premium in this case is 981.2123/4.3612, or \$224.99. The insurance company expects this term policy to be profitable if the annual premium is greater than \$224.99. Put differently, if the company were to charge each of a large number of 60-year-old male clients \$224.99 per annum as a premium for a five-year term policy for \$10,000, then, in the long run, its payments for death claims are expected to just about balance its income from premiums and the interest obtained by investing the premiums at 5%.

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* We should be saying "the expected *average* net discounted income ...", but, in conformity with current usage, we omit the word "average" from now on.

This example illustrates the method used by insurance companies to determine the pure premium of a policy. The actual premium charged will be higher because the company must also cover administration expenses and earn a profit. In practice, the actual premium is obtained by multiplying the pure premium by a mark-up factor. For example, if the mark-up factor is 1.25, the actual premium will be approximately 225×1.25 or \$281.25.

Determining pension payments

The purpose of life insurance is to provide an amount of money to the beneficiary of the policy in the event of the death of the insured. By contrast, an annuity (pension) provides the annuitant with an agreed upon income throughout his or her lifetime. There are many types of annuities, but the principles upon which all are based can be illustrated with the following simple case.

A 60-year-old man about to retire has \$50,000 available from the liquidation of his assets. He may use this sum to purchase an annuity from an insurance company. The company agrees to make a payment at the beginning of each year for as long as the annuitant is alive. When he dies, no further payments are made. The question is: how much should the insurance company agree to pay each year?

Figure 2 shows the payoffs from the point of view of the insurance company; revenue is indicated by a positive, and outlay by a negative sign. P is the annual payment to the annuitant. The first payment is made at the beginning of the second year (or, what amounts to the same thing, at the end of the first year) of the contract.

The annuitant may die at age 60, or 61, or 62, ..., or 109 (we assume that the probability of surviving age 109 is 0). The probabilities of death at ages 60 to 64 for 60-year-old males are obtained from Table 1; the probabilities of death at ages 65 to 109 can be determined from the detailed mortality table. Table 3 shows the probability distribution of the discounted net income of the company.

For example, if the annuitant dies at age 62, the net income of the company consists of the initial receipt of \$50,000 minus two payments of P each at the beginning of the second and third years of the contract. The expected net discounted income of the company is

$$(50,000)(0.0200) + (50,000 - Pv)(0.0214) + ...$$
$$+ [50,000 - P(v + v^2 + ... + v^{49})](0.0001).$$

For a given interest rate (i), the discount factor (v) and its powers can be calculated; the expected net discounted income is of the form $50,000 - cP$, where c is some number. The expected net discounted income is zero if the annual payment equals the break-even or pure payment, $P = 50,000/c$.

To illustrate the calculations, let us consider an even simpler type of annuity contract, similar to the one described earlier except that the annual payment will

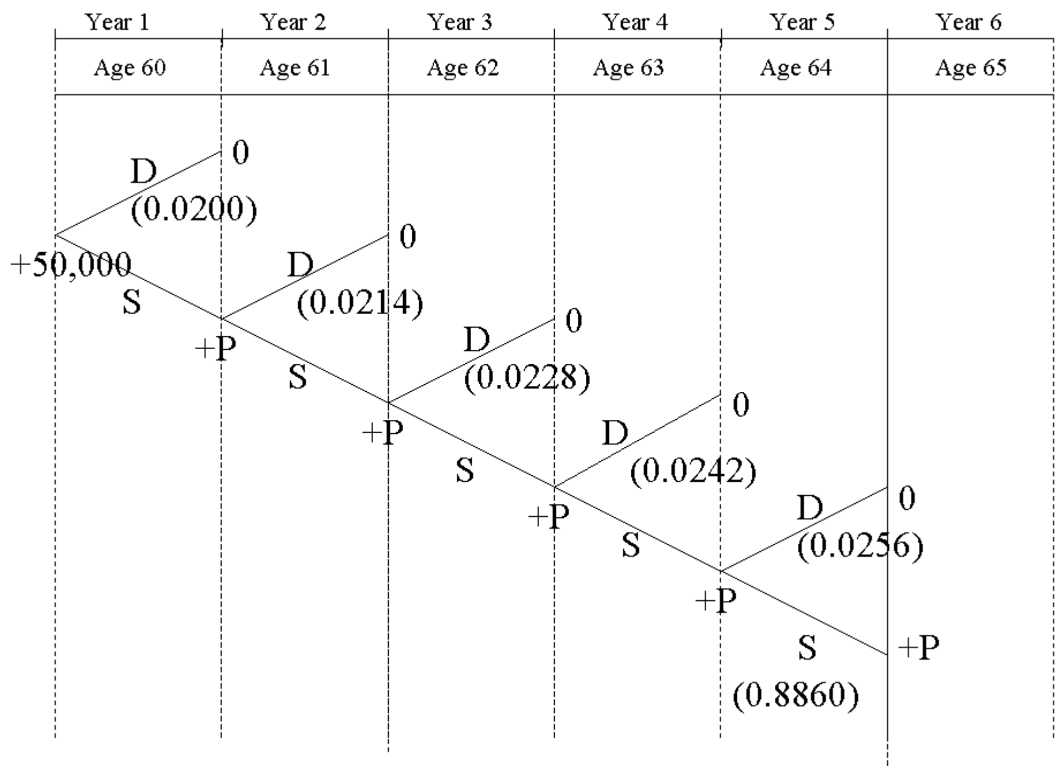

Figure 2
Straight life annuity

be made for as long as the annuitant is alive or for five years, whichever is less. We assume that $i = 0.05$, therefore, $v = 0.9524$, $v^2 = 0.9070$, etc. Table 4 shows the company's discounted net income as a function of the age at death of the annuitant.

For example, if the annuitant dies at age 61 (and the probability of this occurring is 0.0214), the discounted net income is $50,000 - 0.9524P$. The expected net discounted income of the company is

$$(50,000)(0.0200) + (50,000 - 0.9524P)(0.0214) + \cdots + (50,000 - 4.3295P)(0.8860),$$

or,

$$50,000 - [(0.0214)(0.9524) + ... + (0.8860)(4.3295)]P,$$

or

$$50,000 - (4.0554)P.$$

For the expected net discounted income to be positive, the annual payment should not exceed the pure payment of \$12,329.24.

Table 3
Probability distribution of discounted net income,
straight life annuity for \$50,000 starting at age 60

Policy year	Age at death	Probability	Discounted net income
1	60	0.0200	50000
2	61	0.0214	$50000 - Pv$
3	62	0.0228	$50000 - Pv - Pv^2$
4	63	0.0242	$50000 - Pv - Pv^2 - Pv^3$
5	64	0.0256	$50000 - Pv - Pv^2 - Pv^3 - Pv^4$
...	...	...	...
50	109	0.0001	$50000 - P(v + v^2 + ... + v^{49})$
		1.0000	

Table 4
Probability distribution of discounted net income,
straight life annuity for \$50,000 starting at age 60 with a maximum of five payments

Policy year	Age at death	Probability	Discounted net income
1	60	0.0200	50,000
2	61	0.0214	$50,000 - 0.9524P$
3	62	0.0228	$50,000 - 1.8594P$
4	63	0.0242	$50,000 - 2.7232P$
5	64	0.0256	$50,000 - 3.5459P$
	65+	0.8860	$50,000 - 4.3295P$
		1.0000	

In practice, the actual payment will be less than this pure payment, because the insurance company must meet its expenses and be profitable. The actual payment is calculated by applying a markdown to the pure payment. For example, if the markdown is 80%, the actual payment for the above annuity will be \$9,863.39.

This is then, in principle, how annuities are established. As with life insurance, the age at death of the annuitant is uncertain. If the annuitant dies early, the net income of the company is positive; if the annuitant dies late, the net income is negative. By selling a large number of annuities to persons of the same age, the company hopes to offset the losses from some contracts against the gains from others.

PROBLEMS

1: Calculate the pure annual premium of a $20,000, five-year term policy issued to a 60-year-old male. Use Mortality Table 1. Assume that the insurance company can invest its funds at 6%.

2: Calculate the pure annual premium of a $15,000, three-year term policy issued to a 62-year-old male. Use Mortality Table 1. Assume an interest rate of 4%.

3: Calculate the pure annual payment for an annuity purchased by a 60-year-old male with a lump sum of $75,000. The annual payments will be made to the annuitant for life or three years, whichever is less. Use Mortality Table 1. Assume that the insurance company can invest its funds at 6%.

4: The text describes the method used to calculate the annual payment for a "straight" annuity, that is, an annuity that pays an income during a certain period if the annuitant is still alive. (This period could be infinite, so that the payments would be made throughout the annuitant's remaining lifetime.) An *annuity with installments certain* pays an income to the annuitant over a specified period while he/she is alive; if, however, the annuitant dies within a "guarantee period," the payments are continued to a beneficiary selected by the annuitant for the balance of the guarantee period. For example, suppose that a 60-year-old male has $40,000 with which to buy an annuity that will pay P at the end of every year for five years as long as he is still alive, but guarantees that the first two payments will be made whether he is alive or not. Calculate the pure payment assuming that the interest rate is 5%. Use Mortality Table 1.

5: A *cash refund annuity* pays an income to the annuitant over a stipulated period of time if he/she is still alive, but if the annuitant dies before the sum of all income payments received equals or exceeds the money he/she paid, the annuitant's beneficiary receives the balance in one sum.

For instance, suppose that a 60-year-old male can buy for $60,000 a cash refund annuity that will pay $25,000 at the end of every year for three years as long as he is alive. If he dies before the sum of all payments equals or exceeds $60,000, his beneficiary will receive the difference in a lump sum at the end of the year. For example, if he dies during the first policy year, the company will pay $60,000 to his beneficiary at the end of that year. Calculate the expected net discounted income for the insurance company, assuming it can get a 12% rate of return on its investments. Use Mortality Table 1.

6: The caption accompanying an Associated Press photo:

> $1,776 A WEEK FOR LIFE ... A hairstylist Eric L., 26, of North Arlington, N.J., embraces his fiancée, bookkeeper Mathilde C., yesterday after winning $1,776 a week for life in the New Jersey lottery. If he lives to be 76, he will get $4.5 million. If he doesn't, his heirs are guaranteed $1.8 million. He plans to open a youth center.

Suppose you are responsible for determining the expected discounted value of a lottery prize which guarantees the winner $100,000 a year for life or 50 years, whichever is less, and the winner's heirs $2 million if the winner dies before collecting a total of $5 million. The first payment will be made on January 1 when the lottery winners will be drawn, and all subsequent payments will be made on January 1 of the following years.

(a) Without actually doing any calculations, state as precisely as possible how you would determine the expected discounted value of the prize under the assumption that the winner will be a 60-year-old male.

(b) Again without doing any calculations, state precisely how you would determine the expected discounted value of the prize, allowing for the fact that the winner could be male or female and that his or her age next January 1 could vary between 0 and 109. State carefully any assumptions you are forced to make and justify the need for any additional information you may require.

7: Some serial publications sell a life subscription, that is, a subscription that lasts for as long as the subscriber is alive. For example, the cost of a National Geographers' Society Life Membership, which is essentially a life subscription to the *National Geographer*, is $200. The price is the same regardless of the age (provided it is greater than 10) and gender of the subscriber. Assume that you represent the National Geographers' Society. Without actually doing any calculations, state as precisely as possible how you would determine the price of a life subscription to the National Geographer. State carefully and justify any assumptions you are forced to make.

COMPUTING

The computer program **PensIns** calculates the pure premium of whole life, term life, and endowment policies, and the pure payment of term and life annuities, for any age of the insured or annuitant. The program file is named `PensIns.exe`. One of four mortality tables (national and regional, male and female) may be selected. On request, the program also lists the probability distribution of age at death, and/or that of net income or discounted net income. (The mortality tables can be displayed by requesting the probability distribution of the age at death of an insured for whole life insurance or life pension aged 0.)

The following problems require this computer program. Assume throughout that the interest rate is 8%. You may choose either the national or the regional mortality tables provided.

(a) Calculate the pure annual premium of a whole life insurance policy for $100,000 issued to (i) a 25-year old male and (ii) a 25-old female.

(b) Same as (a), except that the policy is a 10-year term insurance policy.

(c) Same as (a), except that the policy is a 15-year term endowment policy.

(d) Calculate the pure annual payments of a whole life annuity purchased with a lump sum of $100,000 by (i) a 65-year old male and (ii) a 65-year old female.

(e) Same as (d), except that the annuity has a term of 10 years.

Using the computer program and either the national or the regional mortality tables, calculate and display graphically:

(f) the relationship between age and pure premium per $1,000 of whole life insurance by gender for a given interest rate;

(g) the relationship between age and pure annual payment per $1,000 of lump-sum purchase payment by gender for a given interest rate.