The Imaginary Number i

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= -1 \end{aligned} \tag{1}$$

By extension

$$\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = 3i$$

A *complex number*, z, has a *real* part and an *imaginary* part: $z = a + bi$, where a and b are real, and $i = \sqrt{-1}$.

Sum of two complex numbers:

$$(a + ib) + (c + id) = (a + c) + (b + d)i$$

Product of two complex numbers:

$$\begin{aligned} (a + ib) \cdot (c + id) &= ac + i(ad + bc) + i^2(bd) \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Why do we need i? Try finding what x might be in the equation

$$x^2 + x + 1 = 0 \tag{2}$$

Here's how NOT to do it:

$$\text{Eqn 2}: \Rightarrow \quad x + 1 = -x^2 \tag{3}$$

$$\text{Eqn 2}: \Rightarrow \quad x(x+1) + 1 = 0 \tag{4}$$

$$\text{Sub (3) in (4)}: \Rightarrow \quad -x^3 + 1 = 0 \tag{5}$$

$$\Rightarrow \quad x = 1 \tag{6}$$

Verify by substituting Eqn (6) into Eqn (2):

$$\begin{aligned} 1^2 + 1 + 1 &= 0 \\ 3 &= 0 \qquad \text{oooops} \ \ldots \end{aligned}$$

Here's something better:

$$
\begin{aligned}
x^2 + x + 1 &= 0 \\
x &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot (1) \cdot (1)}}{2} \\
&= \frac{-1 \pm i\sqrt{3}}{2}
\end{aligned}
$$

For the first root, $(\frac{-1}{2} + i\frac{\sqrt{3}}{2})$, we have:

$$
\begin{aligned}
x^2 &= \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= \left(\frac{1}{4} + \frac{-3}{4}\right) + i\left(\frac{-\sqrt{3}}{4} + \frac{-\sqrt{3}}{4}\right) \\
&= \frac{-1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}
$$

Verify:

$$
\begin{aligned}
x^2 + x + 1 &= \left(\frac{-1}{2} - i\frac{\sqrt{3}}{2}\right) + \left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right) + 1 \\
&= 0
\end{aligned}
$$

Complex numbers can be represented as vectors (or points) in the "complex plane". By convention, the x axis is real, the y axis is imaginary.

Ex.: $z_1 = (2 + i)$,

$$z_2 = (-1 + 3i) \ ,$$

$$z_3 = (3 - 3i), \ \ldots$$

$z_1 + z_2 = 1 + 4i$: there's a simple geometrical relation between the vectors representing z_1, z_2, and $z_1 + z_2$.

By definition, the complex conjugate of a complex number $z = a + ib$, denoted by z^*, is equal to

$$z^* = a - ib$$

that is, we just change the sign of the imaginary part.

Note that $z \cdot z^*$ is *always* a positive real number:

$$\begin{aligned} z \cdot z^* &= (a + ib)(a - ib) \\ &= a^2 - abi + abi - i^2b^2 \\ &= a^2 + b^2 \end{aligned}$$

Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{7}$$

Special case:

$$e^{i\pi} = -1$$

Related formulas:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Physical quantities like mass, energy, time, temperature, electric field, magnetic field, . . . are always real.

However, in writing mathematical relationships between physical quantities, it is often convenient to introduce complex variables.

Sometimes, the real part of a complex variable,
$\mathrm{Re}(a + ib) = a$, has a physical meaning;

Sometimes, the imaginary part of a complex variable,
$\mathrm{Im}(a + ib) = b$, has a physical meaning;

Sometimes, the square of a complex variable,
$zz^* = (a + ib)(a - ib) = a^2 + b^2$, has a physical meaning.

etc.

In these instances, a, b, and $a^2 + b^2$ are real numbers!