

Problem 1. (4 pts)

Show that  $((p \wedge q)) \rightarrow (p \rightarrow q)$  is a tautology :

a) Using a truth table

$p$	$q$	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
F	F	F	T	T
F	T	F	T	T
T	F	F	F	T
T	T	T	T	T



b) Using logical equivalence.

$$\begin{aligned} & (p \wedge q) \rightarrow (p \rightarrow q) \\ \Leftrightarrow & \neg(p \wedge q) \vee (p \rightarrow q) \\ \Leftrightarrow & (\neg p \vee \neg q) \vee (\neg p \vee q) \\ \Leftrightarrow & (\neg p \vee \neg p) \vee (\neg q \vee q) \\ \Leftrightarrow & \neg p \vee T \\ \Leftrightarrow & T \end{aligned}$$

Problem 2. (4 pts) Let  $Q(x, y)$  be the statement " $x + 1 = 2y$ ", where the domain of discourse is the set of all real numbers. State the truth values of

a)  $\forall x \exists y Q(x, y)$

T ✓

b)  $\exists x \forall y Q(x, y)$

F ✓

c) State the converse of  $\neg p \rightarrow q$

$q \rightarrow \neg p$  ✓

d) State the contrapositive of  $\neg p \rightarrow q$

$\neg q \rightarrow \neg \neg p$

$\neg q \rightarrow p$  ✓

Problem 3. (4 pts)

Let  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  be given by  $f(n) = 2\lfloor \frac{n}{2} \rfloor$ .

a) Is  $f$  one-to-one? Justify your answer.

No

$$f(0) = 0$$

$$f(1) = 2 \lfloor \frac{1}{2} \rfloor = 2 \cdot 0 = 0$$

$$\text{So } f(0) = f(1) \text{ but } 0 \neq 1$$

b) Is  $f$  onto? Justify your answer.

No,  $f(n)$  even for any  $n$  So no odd number is in range of  $f$

Problem 4. (4 pts)

Let  $A = \{a, b, c, d\}$ ,  $B = \{b, d, f, k\}$ ,  $C = \{a, g\}$ . Find the following:

a)  $(A \cup B) \cap C$

$$= \{a, b, c, d, f, k\} \cap \{a, g\} = \{a\}$$

b)  $p(C)$  (The power set of  $C$ )

$$\{\emptyset, \{a\}, \{g\}, \{a, g\}\}$$

c)  $A - B$

$$= \{a, b, c, d\} - \{b, d, f, k\}$$

$$= \{a, c\}$$

d)  $C \times A$

$$= \{(a, a), (a, b), (a, c), (a, d), (g, a), (g, b), (g, c), (g, d)\}$$

Problem 5. (4 pts)

Let  $A, B, C$  be arbitrary sets. prove, using the Laws of Sets, that

$$A - (B \cap C) = (A - B) \cup (A - C)$$

$$\begin{aligned} A - (B \cup C) &= A \cap (\overline{B \cup C}) \\ &= A \cap (\overline{B} \cap \overline{C}) \\ &= (A \cap \overline{B}) \cup (A \cap \overline{C}) \\ &= (A - B) \cup (A - C) \end{aligned}$$

Problem 6. (4 pts)

Calculate the following:

a)  $-88 \bmod 13$  ~~ans~~  $-88 = -7 \times 13 + \boxed{3}$   
Ans

b)  $18 \bmod 7$   $18 = 2 \cdot 7 + \boxed{4}$

c) Is  $175 \equiv 22 \bmod 17$ ? Yes  
 $175 - 22 = 153$  and  $17 \mid 153$

d) Use the Euclidean Algorithm to calculate  $\gcd(303, 102)$ .

$$303 = 2 \cdot 102 + 99$$

$$102 = 1 \cdot 99 + \boxed{3}$$

$$99 = 3 \cdot 33 + 0$$

Problem 7. (4 pts)

a) Prove that if  $a \equiv b \pmod{n}$  and  $b \equiv c \pmod{n}$  then  $a \equiv c \pmod{n}$

$$\text{If } a \equiv b \pmod{n} \text{ then } a = b + k_1 n$$

$$\text{If } b \equiv c \pmod{n} \text{ then } b = c + k_2 n$$

$$\therefore a = c + \cancel{k_1 n} + k_2 n + k_1 n$$

$$= c + (k_2 + k_1)n$$

$$\therefore a \equiv c \pmod{n}$$

b) Prove that if  $f : A \rightarrow B$  is one to one and  $g : B \rightarrow C$  is one-to-one, then  $g \circ f : A \rightarrow C$  is one-to-one.

$$\text{If } g \circ f(x) = g \circ f(y)$$

$$\text{Then } g(f(x)) = g(f(y))$$

$$\text{Since } g \text{ 1-1 } \Rightarrow f(x) = f(y)$$

$$\text{Since } f \text{ 1-1 } \Rightarrow x = y$$

$$\therefore g \circ f \text{ is 1-1}$$

Problem 8. ( 4 pts)

a) Find an inverse for  $7 \bmod 17$

$$17 = 2 \cdot 7 + 3$$

$$7 = 2 \cdot 3 + 1$$

$$\begin{aligned} \therefore 1 &= 7 - 2 \cdot 3 \\ &= 7 - 2 \cdot (17 - 2 \cdot 7) \\ &= 5 \cdot 7 - 2 \cdot 17 \end{aligned}$$

Therefore  $\boxed{5}$  is an inverse for  $7 \bmod 17$

b) Find all solutions to

$$3x \equiv 4 \pmod{13}$$

1) Find an inverse of 3 mod 13

$13 = 4 \cdot 3 + 1$  so  $1 = 13 - 4 \cdot 3$  and  $-4$  is an inverse for 3 mod 13

2)  $-4 \cdot 3 x \equiv (-4)(4) \pmod{13}$

But  $-4 \cdot 3 \equiv 1 \pmod{13}$

therefore  $x \equiv -16 \pmod{13}$

3) Any other solution is of the form

$$\boxed{-16 + k \cdot 13}$$