

Problem 1. (4 pts)

Show that $((p \wedge q)) \rightarrow (p \vee q)$ is a tautology :

a) Using a truth table

P	q	$p \wedge q$	$p \vee q$	$p \wedge q \rightarrow p \vee q$
F	F	F	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	T	T

b) Using logical equivalence.

$$\begin{aligned} & (p \wedge q) \rightarrow (p \vee q) \\ \Leftrightarrow & \neg(p \wedge q) \vee (p \vee q) \\ \Leftrightarrow & (\neg p \vee \neg q) \vee (p \vee q) \\ \Leftrightarrow & (\neg p \vee p) \vee (\neg q \vee q) \\ \Leftrightarrow & T \vee T \\ \Leftrightarrow & T \end{aligned}$$

Problem 2. (4 pts) Let $Q(x, y)$ be the statement " $x + 1 = 2y$ ", where the domain of discourse is the set of all real numbers. State the truth values of

a) $\exists y \forall x Q(x, y)$

F

b) $\forall y \exists x Q(x, y)$

T

c) State the converse of $p \rightarrow \neg q$

$\neg q \rightarrow p$

d) State the contrapositive of $p \rightarrow \neg q$

$\neg p \rightarrow q$

or $\neg q \rightarrow \neg p$

Problem 3. (4 pts)

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be given by $f(n) = \lceil \frac{n}{2} \rceil$.

a) Is f one-to-one? Justify your answer.

$$\begin{aligned}\underline{\text{No}} \quad f(1) &= \lceil \frac{1}{2} \rceil = 1 \\ f(2) &= \lceil \frac{2}{2} \rceil = 1 \\ \text{So } f(1) &= f(2) \text{ but } 1 \neq 2\end{aligned}$$

b) Is f onto? Justify your answer.

$$\begin{aligned}\underline{\text{Yes}} \quad \text{Given any integer } k, \\ f(2k) &= \lceil \frac{2k}{2} \rceil = \lceil k \rceil = k \\ \text{So } f \text{ is } \underline{\text{onto}}\end{aligned}$$

Problem 4. (4 pts)

Let $A = \{a, c, e, h\}$, $B = \{c, g\}$, $C = \{a, e, g, l\}$. Find the following:

a) $(A \cup B) \cap C$

$$\begin{aligned} &= \{a, c, e, h, g\} \cap \{a, e, g, l\} \\ &= \{a, e, g\} \end{aligned}$$

b) $\wp(B)$ (The power set of B)

$$\{\emptyset, \{c\}, \{g\}, \{c, g\}\}$$

c) $A - B$

$$\begin{aligned} &= \{a, c, e, h\} - \{c, g\} \\ &= \{a, e, h\} \end{aligned}$$

d) $B \times C$

$$= \{(c, a), (c, e), (c, g), (c, l), (g, a), (g, e), (g, g), (g, l)\}$$

Problem 5. (4 pts)

Let A, B, C be arbitrary sets. prove, using the Laws of Sets, that

$$\begin{aligned} A - (B \cap C) &= (A - B) \cup (A - C) \\ A - (B \cap C) &= A \cap (\overline{B \cap C}) \\ &= A \cap (\overline{B} \cup \overline{C}) \\ &= (A \cap \overline{B}) \cup (A \cap \overline{C}) \\ &= (A - B) \cup (A - C) \end{aligned}$$

Problem 6. (4 pts)

Calculate the following:

a) $-62 \bmod 15$

$$-62 = -5 \cdot 15 + \boxed{13}$$

b) $23 \bmod 6$

$$23 = 3 \cdot 6 + \boxed{5}$$

c) Is $199 \equiv 7 \bmod 17$?

No, $199 - 7 = 192$ and $17 \nmid 192$

d) Use the Euclidean Algorithm to calculate $\gcd(302, 201)$.

$$302 = 1 \cdot 201 + 101$$

$$201 = 1 \cdot 101 + 100$$

$$101 = 1 \cdot 100 + \boxed{1}$$

$$100 = 1 \cdot 100 + 0$$

Problem 7. (4 pts)

a) Prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a - c \equiv b - d \pmod{n}$

$$\text{If } a \equiv b \pmod{n} \text{ then } a = b + k_1 n$$

$$\text{If } c \equiv d \pmod{n} \text{ then } c = d + k_2 n$$

$$\begin{aligned}\therefore a - c &= b + k_1 n - (d + k_2 n) \\ &= b - d + (k_1 - k_2)n\end{aligned}$$

$$\therefore a - c \equiv b - d \pmod{n}$$

b) Prove that if $f : A \rightarrow B$ is onto and $g : B \rightarrow C$ is onto, then $g \circ f : A \rightarrow C$ is onto.

Let $c \in C$. Since g is onto, there exists $b \in B$ with $g(b) = c$.

Since f is onto, there exists $a \in A$ with $f(a) = b$.

$$\begin{aligned}\text{Thus } g \circ f(a) &= g(f(a)) \\ &= g(b) \\ &= c\end{aligned}$$

Therefore $g \circ f$ is onto.

Problem 8. (4 pts)

a) Find an inverse for $7 \bmod 19$

$$19 = 2 \cdot 7 + 5$$

$$7 = 1 \cdot 5 + 2$$

$$5 = 2 \cdot 2 + 1$$

$$\begin{aligned}\therefore 1 &= 5 - 2 \cdot 2 \\&= 5 - 2(7 - 5) \\&= 3 \cdot 5 - 2 \cdot 7 \\&= 3(19 - 2 \cdot 7) - 2 \cdot 7 \\1 &= 3 \cdot 19 - 8 \cdot 7\end{aligned}$$

Therefore $\boxed{-8}$ is an inverse for $7 \bmod 19$

b) Find all solutions to

$$5x \equiv 3 \pmod{16}$$

1) Find an inverse for $5 \pmod{16}$

$$16 = 3 \cdot 5 + 1$$

$\therefore 1 = 16 - 3 \cdot 5$ and -3 is an inverse for
 $5^{-1} \pmod{16}$

2) $-3 \cdot 5x \equiv (-3)(3) \pmod{16}$

But $-3 \cdot 5 \equiv 1 \pmod{16}$

$\therefore x \equiv -9 \pmod{16}$

3) Any other solution is of the form

$$\boxed{-9 + k \cdot 16} \quad k \text{ integer}$$