Problem Solutions

8.3 5) $\{1, \sqrt{2}\}$ basis for $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$

$\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2\}$ basis for $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}(\sqrt{2})$

So $\{1, \sqrt[3]{2}, (\sqrt[3]{2})^2, \sqrt{2}, \sqrt{2}\sqrt[3]{2}, \sqrt{2}(\sqrt[3]{2})^2\}$ is a basis of $\mathbb{Q}(\sqrt{2}, \sqrt[3]{2})$ over $\mathbb{Q}$ and $[\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}) : \mathbb{Q}] = 2 \cdot 3 = 6$

21) If E is a finite extension of F, every $\alpha \in E, \alpha \notin F$ is algebraic over F, and $\deg(\alpha, F) \mid [E : F]$, But $\deg(\alpha, F) = [F(\alpha) \mid F]$ so $[F(\alpha) : F] \mid [E : F]$, which is prime and $[F(\alpha) : F] > 1$ since $\alpha \notin F$. Hence $[F(\alpha) : F] = [E : F] \Rightarrow F(\alpha) = E$

8.5 5) If a is a generator of the 18^{th} roots of unity then any other generator is of the form a^n, where n is relatively prime to 18. So the number of primitive 18^{th} roots is $\varphi(18) = |\{1, 5, 7, 11, 13, 17\}| = 6$

11) Since α generates F^*, and $\mathbb{Z}_p(\alpha) \subseteq F$, in fact $\mathbb{Z}_p(\alpha) = F$

Thus $F = \{a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1}\}$ by a counting argument $(|F| = p^n)$

So $\{1, \alpha, \cdots \alpha^{n-1}\}$ is a basis for F over $\mathbb{Z}_p(\alpha)$, and $\alpha^n = b_0 + b_1\alpha + \cdots + b_{n-1}\alpha^{n-1}$ and α satisfies $p(x) = b_0 + b_1 x + \cdots + b_{n-1}x^{n-1} - x^n$

Moreover, α can't satisfy any lower degree polynomial by L.I. of $\{1, \alpha, \cdots, \alpha^{n-1}\}$ so $p(x)$ is $\operatorname{irr}(\alpha, F)$ and $\deg(\alpha, F) = n$

9.1 15) $x \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) \Rightarrow x = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, $a, b, c, d \in \mathbb{Q}$

$\sigma_1(x) = a - b\sqrt{2} + c\sqrt{3} - d\sqrt{6}$ $\quad \sigma_1(x) = x \Rightarrow b = d = 0$

$\sigma_2(x) = a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6}$ $\quad \sigma_2(x) = x \Rightarrow c = d = 0$

$\sigma_3(x) = a - b\sqrt{2} - c\sqrt{3} + d\sqrt{6}$ $\quad \sigma_3(x) = x \Rightarrow b = c = 0$

$E_{\{\sigma_1, \sigma_3\}} = \mathbb{Q}$, $\quad E_{\{\sigma_3\}} = \mathbb{Q}(\sqrt{6})$ $\quad E_{\{\sigma_2, \sigma_3\}} = \mathbb{Q}$