Prob 1)

1) F (Z not a field)

2) F $\mathbb{Z}_2 \times \mathbb{Z}_4$ not cyclic, $\mathbb{Z}_8$ is

3) F Converse of Lagrange's Thm false

4) T Product defined by multiplying rep$^{\triangle}$

5) F $S_3/A_3 \cong \mathbb{Z}_2$ abelian but S_3 is not

Prob 2 $36 = 4 \cdot 9 = 2^2 \cdot 3^2$

a) $2 - \{2\}, \{1,1\}$, $2 - \{2\}, \{1,1\}$

$\mathbb{Z}_{2^2} \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_{2^2} \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{3^2}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

or $\mathbb{Z}_4 \times \mathbb{Z}_9 \cong \mathbb{Z}_{36}$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

b) Each of these groups is also a ring. Calculate the order of the multiplicative identity in the ring

$\mathbb{Z}_4 \times \mathbb{Z}_9$ $\quad \operatorname{Lcm}(4,9) = 36$

$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ $\quad \operatorname{Lcm}(4,3,3) = 12$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9$ $\quad \operatorname{Lcm}(2,2,9) = 18$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ $\quad \operatorname{Lcm}(2,2,3,3) = 6$

Prob 3)

a) N is a subring $\ni$

$aN \subseteq N, \quad Na \subseteq N \quad \forall a \in R$

b) $\langle 3 \rangle$ is a subgroup, hence closed

So $n \cdot a = \underbrace{a + \cdots + a}_{n} \in \langle 3 \rangle \quad \forall n \in \mathbb{Z}_{12},\ a \in \langle 3 \rangle$

Also $a \cdot n = n \cdot a \in \langle 3 \rangle \quad \forall n \in \mathbb{Z}_{12},\ a \in \langle 3 \rangle$

So $\langle 3 \rangle$ is an ideal

Cosets $\quad \bar{0} = \langle 3 \rangle$

$\bar{1} = 1 + \langle 3 \rangle$

$\bar{2} = 2 + \langle 3 \rangle$

+	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{0}$
$\bar{2}$	$\bar{2}$	$\bar{0}$	$\bar{1}$

x	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{1}$

Prob 4) a) $Z(G)$ subgroup - see model sol

is $g \in Z(G)$ $\quad gxg^{-1} = x \quad \forall x \in G$

$\Rightarrow \quad gx = xg \Rightarrow g = xgx^{-1} \quad \forall x \in G$

$\Rightarrow \quad Z(G)$ normal

b) Assume $G/Z(G)$ ~~normal~~ cyclic

Let $aZ(G)$ generate $G/Z(G)$

Then if $g_1, g_2 \in G$,

$g_1 = a^k h_1, \quad g_2 = a^j h_2 \qquad h_1, h_2 \in Z(G)$

$g_1 g_2 = a^k h_1 a^j h_2 = a^k a^j h_1 h_2 = a^j a^k h_2 h_1 = a^j h_2 a^k h_1$

$= g_2 g_1$

5) If G transitive, $x \in X$ then

$\forall x_1 \in X$, $\exists g_1$ s.t $g_1 x = x_1$

So $\{g \cdot x \mid g \in G\} = X$ and G has a single orbit

Conversely, if $\exists x_0 \in X \ni \{g \cdot x_0 \mid g \in G\} = X$ then $\forall x_1, x_2 \in X$ $\exists g_1, g_2 \ni$

$g_1 x_0 = x_1$
$g_2 x_0 = x_2$ $\Rightarrow g_2 g_1^{-1} x_1 = x_2$ so G is transitive

6) If ϕ is a homomorphism of G int G' with kernel $\ker\phi$, then

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "\nu"'] & \phi(G) \subseteq G' \\ G/\ker\phi \arrow[ur, "\mu"', "\cong"] & \end{tikzcd}$$

ν cannonical hom. morphism $\nu(g) = g\ker\phi$, μ isomorphism $\mu(g\ker\phi) = \phi(g)$ and $\phi = \mu \circ \nu$

Ex $G = S_3$, $\ker\phi = A_3$

$$\begin{tikzcd} S_3 \arrow[r, "\phi"] \arrow[d] & \{\pm1\} \\ S_3/A_3 \arrow[ur, phantom, "\cong"] & \end{tikzcd}$$