Ma 3020 Term Test 3 Solutions

1) Units in $\mathbb{Z}_9 = \{1,2,4,5,7,8\}$, 2 is a generator since
$2^1, 2^2=4, 2^3=8, 2^4\equiv 7, 2^5\equiv 5, 2^6\equiv 1$

2) a) $24^{21} = 24^{3\cdot 6+3} \equiv 3^{3\cdot 6+3} \equiv (3^6)^3\cdot 3^3 \equiv 1^3\cdot 3^3 \equiv 27 \equiv \underline{6 \bmod 7}$

b) $\gcd(15,12)=3$ and $3 \mid 9$ so $\exists$ a solution
reduced equation: $5x \equiv 3 \bmod 4$ but $5\equiv 1 \bmod 4$ so $x\equiv 3 \bmod 4$
Solutions $\{ \underline{3+n12},\ \underline{7=3+4 \bmod 12},\ \underline{11=3+2\cdot 4 \bmod 12}$

3) 1) F (since degree of $f(x)$ may be >3)
2) F (require $p \nmid a$)
3) T (By Eisensteins Criteria, with $p=3$)
4) F (ideal must be of the form $\langle f(x)\rangle$, $f(x)$ irreducible in $F[x]$)
5) T (x^2-6+6 irreducible over $\mathbb{Q}$)

4) a) $\underline{c=2}$

b) Let α be a zero of x^3+x^2+2; $\alpha^3 = -\alpha^2-2 = 2\alpha^2+1$
$F(\alpha) = \{a+b\alpha+c\alpha^2 \mid a,b,c \in \mathbb{Z}_3\}$ with addition mod 3, multiplication mod 3, and products of degree higher than 2 reduced using $\alpha^3 = 2\alpha^2+1$

c) $(\alpha^2+1)(\alpha^2+2) = \alpha^4+3\alpha^2+2 = \alpha^4+2$
$= \alpha(\alpha^3)+2 = \alpha(2\alpha^2+1)+2$
$= 2\alpha^3+\alpha+2$
$= 2(2\alpha^2+1)+\alpha+2$
$= 4\alpha^2+\alpha+4 = \underline{\alpha^2+\alpha+1}$

5) Every ideal of $F[x]$ is principal, hence of the form $\langle f(x)\rangle$ for some $f(x)\in F[x]$. Suppose $\langle f(x)\rangle \neq \{0\}$ is a proper prime ideal of $F[x]$. We show $f(x)$ irreducible, hence $\langle f(x)\rangle$ is maximal.

If not, $f(x) = g(x)h(x)$ where degrees of $g(x), h(x)$ are $< \deg f(x)$. But then $g(x)h(x) = f(x) \in \langle f(x)\rangle$, and since $\langle f(x)\rangle$ is a prime ideal, either $g(x)\in\langle f(x)\rangle$ or $h(x)\in\langle f(x)\rangle$. But every polynomial in $\langle f(x)\rangle$ is a multiple of $f(x)$, so has degree $\geq \deg f(x)$ which contradicts our assumption that degrees of $g(x), h(x)$ are $< \deg f(x)$

Hence $f(x)$ irreducible so $\langle f(x)\rangle$ is maximal