Ma 3020 Sample Question for Test 2
Solutions

1 a) I_g is a homomorphism:

$$I_g(g_1g_2) = g(g_1g_2)g^{-1} = (gg_1(g^{-1}g)(g_2g^{-1})$$
$$= (gg_1g^{-1})(gg_2g^{-1})$$
$$= I_g(g_1) \circ I_g(g_2)$$

I_g 1-1:

$$I_g(h) = I_g(k)$$
$$\Rightarrow ghg^{-1} = gkg^{-1}$$
$$\Rightarrow hg^{-1} = kg^{-1}$$
$$\Rightarrow h = k$$

I_g onto:

Given $h \in G$, set $k = g^{-1}hg$

Then

$$I_g(k) = g(g^{-1}hg)g^{-1}$$
$$= (gg^{-1})h(gg^{-1})$$
$$= ehe = h$$

b) By a) $I_g(H)$ is a subgroup of the same order as H, so, by hypothesis must be H.

Thus $I_g(H) = H \Rightarrow gHg^{-1} = H \ \forall g \in G$ so H is normal //

2) a) $Z(G)$ is a subgroup

$Z(G)$ is closed:

if $g_1, g_2 \in Z(G)$ then

$$(g_1g_2)h(g_1g_2)^{-1} = g_1 \ g_2hg_2^{-1}g_1^{-1}$$
$$= g_1hg_1^{-1} = h$$

so $g_1g_2 \in Z(G)$

$e \in Z(G)$:

$ehe^{-1} = ehe = h$
$\forall h \in G$

if $g \in Z(G)$, so is g^{-1}

$$ghg^{-1} = h$$
$$\Rightarrow hg^{-1} = g^{-1}h$$
$$\Rightarrow h = g^{-1}hg \ \forall h$$

So $g^{-1} \in Z(G)$

$Z(G)$ normal:

Let $g \in Z(G)$, $h \in G$. Then $ghg^{-1} = h \Rightarrow gh = hg \Rightarrow h^{-1}gh = g \in Z(G)$
$\forall h \in G$

$\therefore h^{-1}Z(G)h \subseteq Z(G)$

So $Z(G)$ is normal

b) If G is abelian, $Z(G) = G$

If G simple, since $Z(G)$ is normal it follows that either $Z(G) = \{e\}$ or $Z(G) = G$. But if G non abelian, $Z(G) \neq G$ so $Z(G) = \{e\}$.

3) a) See text

b) An ideal must be a subgroup of $\langle \mathbb{Z}_6, + \rangle$ so possible ideals are

$\{0\}$, $\{0,2,4\}$, $\{0,3\}$, $\mathbb{Z}_6$

But inspection if I is any of the above, and $a \in \mathbb{Z}_6$, then $aI \subseteq I$, $(Ia \subseteq I)$ so each of the above is an ideal.

$\mathbb{Z}_6/\{0\} \cong \mathbb{Z}_6$, $\mathbb{Z}_6/\{0,2,4\} \cong \mathbb{Z}_2$, $\mathbb{Z}_6/\{0,3\} \cong \mathbb{Z}_3$, $\mathbb{Z}_6/\mathbb{Z}_6 \cong \{0\}$

4) a) if $1 \in N$, then since N is ideal $a = a \cdot 1 \in N \ \forall a \in R$ $\Rightarrow R \subseteq N \Rightarrow R = N$

b) if N is an ideal in $\mathfrak{F}$, $N = \{0\}$,

let $b \in N$, $b \neq 0$. Since $\mathfrak{F}$ is a field, b has an inverse b^{-1} since N is an ideal $1 = b^{-1} \cdot b \in N$. By part a) $N = \mathfrak{F}$.

5) $(g I_{x_1} g^{-1}) x_2 = g I_{x_1} g^{-1} x_2 = g I_x \cdot x_1 = g x_1 = x_2 \Rightarrow g I_{x_1} g^{-1} \subseteq I_{x_2}$

Conversely, if $h \in I_{x_2}$, set $h' = g^{-1} h g$

Then $h' x_1 = g^{-1} h g x_1 = g^{-1} h x_2 = g^{-1} x_2 = x_1$ so $h' \in I_{x_1}$ and so $h = g h' g^{-1} \in g I_{x_1} g^{-1} \Rightarrow I_{x_2} \subseteq g I_{x_1} g^{-1}$

$\therefore I_{x_2} = g I_{x_1} g^{-1}$

6)
1) True $(g_1, g_2) \to (g_2, g_1)$ is an isomorphism
2) False ($\mathbb{Z}_3 \times \mathbb{Z}_8$ is abelian, S_4 is not)
3) True (Basically this is Lagrange $|G| = |H| \times$ (no of left cosets of H))
4) True (If H a proper subgroup, $|H| \mid pq$. Since p, q are prime either $|H| \mid p$ or $|H| \mid q$. In either case $|H|$ is prime, hence H is cyclic
5) True ($G \to \{0'\}$ is the trivial homomorphism of G into G')