Sample Questions - Test 3

1) Units in $\mathbb{Z}_{12}$ = non-zero divisors = $\{m<12 \mid (m,12) \text{ relatively prime}\}$
$= \{1,5,7,11\}$

$\cdot$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Since $5^2=7^2=11^2=1$ This is Klein 4-group

2) $\gcd(36,24)=12$ Since $12 \nmid 15$ there are no solutions

3) $\varphi(24) = |\{m<24 \mid m \text{ relatively prime to } 24\}| = |\{1,5,7,11,13,17,19,23\}| = 8$

So $7^8 \equiv 1 \bmod 24$

$7^{98} = 7^{12\cdot 8+2} = (7^8)^{12}\cdot 7^2 \equiv 1^{12}\cdot 7^2 \equiv 49 \equiv 1 \bmod 24$

4) a) if M maximal in R, R/M is a field, hence an integral domain, so M prime

b) if M prime, $\mathbb{Z}_6/M$ is a finite integral domain, hence a field
So M maximal

c) maximal ideals are
$\langle 2\rangle = \{0,2,4\}$, $\langle 3\rangle = \{0,3\}$ Since $\mathbb{Z}_6/\langle 2\rangle \cong \mathbb{Z}_2$, $\mathbb{Z}_6/\langle 3\rangle \cong \mathbb{Z}_3$ are fields

5) a) $f(x) = x^3+x^2+1$; $f(0)=1$, $f(1)=1$ So the quadratic $f(x)$ has no zeros in $\mathbb{Z}_2$, hence is irreducible

b) $F[x]/\langle x^3+x^2+1\rangle$ is a field, isomorphic to $\{a\alpha^2+b\alpha+c \mid a,b,c\in \mathbb{Z}_2\}$
where $\alpha^3+\alpha^2+1=0$ and addition and multiplication is done by reducing products using $\alpha^3 = -\alpha^2-1 = \alpha^2+1$, and further reducing mod 2
This is a field of 8 elements

6) 1) T

2) F (if $\gcd(a,n)>1$, $\gcd(b,n)>1$, then $\gcd(ab,n)>1$ so ab not a unit

3) F ($f(x) = x^2+3$ $f(1)=4$, $f(2)=0$ so since $f(x)$ quadratic, it is reducible)

4) T

5) F $(x^2-5x+6) = (x-2)(x-3)$ reducible, so $\langle x^2-5x+6\rangle$ not maximal, so $\mathbb{Q}[x]/\langle x^2-5x+6\rangle$ not a field