Complex Hadamard Matrices and Combinatorial Structures

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Abstract

Forty years ago, Goethals and Seidel showed that if the adjacency algebra of a strongly regular graph X contains a Hadamard matrix then X is of Latin square type or of negative Latin square type [8]. We extend their result to complex Hadamard matrices and find only three additional families of parameters for which the strongly regular graphs have complex Hadamard matrices in their adjacency algebras.

We see that there are only three distance regular covers of complete graphs that give complex Hadamard matrices.

1 Introduction

An $n \times n$ complex matrix W is *type II* if

$$\sum_{x=1}^{n} \frac{W(a,x)}{W(b,x)} = \begin{cases} n & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We say W' is *type II equivalent to* W if $W' = MWN$ for some invertible monomial matrices M and N.

Type II matrices were introduced by Sylvester as inverse orthogonal matrices in [13]. In [11], Jones defined spin models to be matrices that satisfy three types of conditions corresponding to the three Reidemeister moves on link diagrams. Spin models give link invariants. For example, the Potts model is the $n \times n$ matrix

$$-u^3 I + u^{-1}(J - I),$$

where $(u^2+u^{-2})^2 = n$ and J is the matrix of all ones, and it gives evaluations of Jones polynomial. Condition (1) corresponds to the second Reidemeister move, hence the term type II matrices.

A connection to combinatorics comes in Nomura's construction of association schemes from type II matrices. Further, type II matrices arise from various interesting combinatorial objects [2].

If all entries of an $n \times n$ type II matrix W has absolute value 1, then (1) is equivalent to

$$W\overline{W}^T = nI.$$

In this case, W is a *complex Hadamard matrix*. Two complex Hadamard matrices are *equivalent* if they are type II equivalent.

Apart from being a generalization of (real) Hadamard matrices, complex Hadamard matrices have applications in quantum information theory, operator theory and combinatorics, see [14], [15] and [16]. We are motivated to construct infinite families of complex Hadamard matrices, and to classify the complex Hadamard matrices of small sizes.

An $mn \times mn$ complex Hadamard matrix is of *Diţă-type* if it is equivalent to a complex Hadamard matrix of the form

$$\begin{pmatrix} U(1,1)V_1 & U(1,2)V_2 & \dots & U(1,n)V_n \\ U(2,1)V_1 & U(2,2)V_2 & \dots & U(2,n)V_n \\ \vdots & \vdots & \ddots & \vdots \\ U(n,1)V_1 & U(n,2)V_2 & \dots & U(n,n)V_n \end{pmatrix}$$

for some $n \times n$ complex Hadamard matrix U and $m \times m$ complex Hadamard matrices $V_1, \dots, V_n$, see [5]. This construction is a special case of the generalized tensor product of type II matrices $(U_1, \dots, U_m) \otimes (V_1, \dots, V_n)$ defined by Hosoya and Suzuki [9]. Diţă's construction gives most of the parametric families of complex Hadamard matrices in the catalogue of small complex Hadamard matrices [16]. To compile the catalogue, it is useful to know if a complex Hadamard matrix if of Diţă-type.

Goethals and Seidel determined the parameters of strongly regular graphs that contain (real) Hadamard matrices in their adjacency algebras - they are of Latin square type and of negative Latin square type. In this paper, we show that there are only three more families of parameters for which the strongly regular graphs give complex Hadamard matrices. The Seidel matrices of these graphs are either regular two-graphs or the neighbourhoods of regular two-graphs. The situation is different for for distance regular covers of complete graphs. There are only finitely many such graphs that give complex Hadamard matrices.

2 Nomura Algebra

Let W be an $n \times n$ type II matrix. For $a, b = 1, \dots, n$, define vector $Y_{a,b}$ by

$$Y_{a,b}(x) = \frac{W(x,a)}{W(x,b)} \quad \text{for } x = 1, \dots, n. \tag{2}$$

The *Nomura algebra* of W is the set

$$\mathcal{N}_W = \{M : Y_{a,b} \text{ is an eigenvector of } M, \text{ for } a, b = 1, \dots, n\}.$$

If follows immediately that $\mathcal{N}_W$ contains I. The type II condition (1) of W implies $J \in \mathcal{N}_W$. So the dimension of $\mathcal{N}_W$ is at least two. Jaeger et al. [10] showed that $\mathcal{N}_W$ is a commutative matrix algebra that is also closed under the entrywise product and transpose. Any algebra that satisfies all these conditions has a basis of 01-matrices $\mathcal{A} = \{A_0, A_1, \dots, A_d\}$ satisfying

1. $A_0 = I$.
2. $\sum_{i=0}^{d} A_i = J$.
3. $A_i^T \in \mathcal{A}$, for $i = 0, \dots, d$.
4. $A_i A_j$ lies in the span of $\mathcal{A}$, for $i, j = 0, \dots, d$.
5. $A_i A_j = A_j A_i$, for $i, j = 0, \dots, d$.

We call $\mathcal{A}$ an *association scheme with d classes* and the span of $\mathcal{A}$ the *Bose-Mesner algebra* of $\mathcal{A}$. When $d = 1$, $\mathcal{A} = \{I, J - I\}$ and we say its Bose-Mesner algebra is *trivial*. When $d = 2$ and A_1 is symmetric, $\mathcal{A} = \{I, A(X), A(\overline{X})\}$ for some strongly regular graph X.

2.1 Theorem. *[10] If W is a type II matrix then $\mathcal{N}_W$ is the Bose-Mesner algebra of an association scheme.* □

Here is a result about the Nomura algebras of equivalent type II matrices.

2.2 Theorem. *[10] Suppose $W_1 = P_1 D_1 W D_2 P_2$, where D_1 and D_2 are invertible diagonal matrices, and P_1 and P_2 are permutation matrices. Then*

$$\mathcal{N}_{W_1} = P_1 \mathcal{N}_W P_1^T.$$

□

We see that the dimension of $\mathcal{N}_W$ is an invariant for equivalent complex Hadamard matrices. In particular, we have this following observation [1].

2.3 Proposition. *Let W be a complex Hadamard matrix. If $\mathcal{N}_W$ is trivial then W is not of Diţă-type.* $\square$

A type II matrix W with two distinct entries a and b, where $a \neq \pm b$, is a linear combination of J and the incidence matrix of a symmetric design [2]. Szöllősi [15] showed that the Hadamard designs and the Menon designs are the only ones that give complex Hadamard matrices. We know that $\mathcal{N}_W$ is trivial if the symmetric design has at least four points [2].

2.4 Proposition. *Suppose W is a complex Hadamard matrix of the form $(t-1)N+J$, where N is the incidence matrix of a symmetric (n,k,λ)-design. If $n > 3$ and $t \neq -1$, then W is not of Diţă-type.* $\square$

Suppose C is a generalized conference matrix with minimal polynomial $z^2 - \beta z - (n-1)$. When t satisfies $t + t^{-1} + \beta = 0$, $tI + C$ is a type II matrix. If follows that $|t| = 1$ if and only if $\beta \in \mathbb{R}$ and $|\beta| \leq 2$. Observe that a symmetric generalized conference matrix is a regular two-graph. Sankey [12] showed that for a regular two-graph C, the type II matrix $tI + C$ has trivial Nomura algebra if $\beta \neq \pm 2$.

2.5 Proposition. *Let C be a regular two-graph with minimal polynomial $z^2 - \beta z - (n-1)$. If $|\beta| < 2$ and $t + t^{-1} + \beta = 0$ then the complex Hadamard matrix $tI + C$ is not of Diţă-type.* $\square$

3 Strongly Regular Graphs

A *strongly regular graph* with parameters (v, k, a, c) is a k-regular graph X, that is not complete, in which every pair of adjacent vertices have a common neighbours and every pair of distinct non-adjacent vertices have c common neighbours.

The adjacency matrix of X, $A(X)$, satisfies

$$A(X)^2 - (a-c)A(X) - (k-c)I = cJ.$$

Observe that $A(X)$ is the incidence matrix of a symmetric (v, k, c)-design when $a - c = 0$.

Since X is k-regular, k is an eigenvalue of $A(X)$ with eigenvector $\mathbf{1}$. Let θ and τ be the other two eigenvalues of $A(X)$. The eigenvectors of $A(X)$ for θ and τ are orthogonal to $\mathbf{1}$, hence θ and τ are the roots of

$$z^2 - (a-c)z - (k-c) = 0.$$

We conclude that $\theta\tau = c - k$ is non-zero when $c < k$, that is when X is not isomorphic to the complement of mK_{k+1}, for some integer $m \geq 2$. If X is not isomorphic to mK_{k+1} nor its complement then, without loss of generality, we have $k > \theta > 0 > \tau$. Further, if X is not a conference graph then θ and τ are integers, see [6].

Note that $\overline{X}$ is a strongly regular graph with parameters $(v, v-k-1, v-2-2k+c, v-2k+a)$ and $A(\overline{X}) = J - I - A(X)$ has eigenvalues $(v-k-1), (-1-\tau)$ and $(-1-\theta)$.

From the quadratic above, we see that

$$\begin{aligned} c &= k + \theta\tau, \\ a &= k + \theta\tau + \theta + \tau, \quad \text{and} \\ v &= \frac{(k-\theta)(k-\tau)}{k+\theta\tau}. \end{aligned}$$

The multiplicity of θ and τ are

$$m_\theta = \frac{(v-1)\tau + k}{\tau - \theta} \quad \text{and} \quad m_\tau = \frac{(v-1)\theta + k}{\theta - \tau}.$$

respectively. The integrality of m_θ and m_τ impose a necessary condition on (v, k, a, c) for the existence of the strongly regular graph [6].

We list all type II matrices in the Bose-Mesner algebra of X.

3.1 Theorem. *[2] Let X be a strongly regular graph with $v > 4$ vertices, valency k, and eigenvalues k, θ and τ, where $\theta \geq 0 > \tau$. Suppose*

$$W := I + xA(X) + yA(\overline{X}).$$

Then W is a type-II matrix if and only if one of the following holds

(a) $y = x = \frac{1}{2}(2 - v \pm \sqrt{v^2 - 4v})$ *and W is the Potts model.*

(b) $x = \frac{1}{2}(2 - v \pm \sqrt{v^2 - 4v})$ *and $y = (1 + xk)/(x + k)$, and X is isomorphic to mK_{k+1} for some $m > 1$.*

(c) $x = 1$ *and*

$$y = \frac{v + 2(1+\theta)(1+\tau) \pm \sqrt{v^2 + 4v(1+\theta)(1+\tau)}}{2(1+\theta)(1+\tau)},$$

and $A(\overline{X})$ *is the incidence matrix of a symmetric* $(v, v-k-1, \lambda)$*-design where* $\lambda = v - k - 1 + (1+\theta)(1+\tau))$.

(d) $x = -1$ *and*

$$y = \frac{-v + 2\theta^2 + 2 \pm \sqrt{(v-4)(v-4\theta^2)}}{2(1+\theta)(1+\tau)},$$

and $A(X)$ *is the incidence matrix of a symmetric* $(v, k, k+\theta\tau)$*-design.*

(e) $x + x^{-1}$ *is a zero of the quadratic*

$$-\theta\tau z^2 + \alpha z + \beta + 2\theta\tau \tag{3}$$

with

$$\alpha = v(\theta + \tau + 1) + (\theta + \tau)^2,$$
$$\beta = v + v(1 + \theta + \tau)^2 - 2\theta^2 - 2\theta\tau - 2\tau^2$$

and

$$y = \frac{\theta\tau x^3 - [v(\theta+\tau+1) - 2\theta - 2\tau - 1]x^2 - (v + 2\theta + 2\tau + \tau\theta)x - 1}{(x^2-1)(1+\theta)(1+\tau)}.$$

□

In the first two cases, $|x| \neq 1$ for $v > 4$. In case (c), $A(\overline{X})$ is the incidence matrix of a symmetric design, hence $(-1-\theta) + (-1-\tau) = 0$. In Case (d), $A(X)$ is the incidence matrix of a symmetric design, so $\theta + \tau = 0$. We have the following lemma for the all cases.

3.2 Lemma. *Suppose* X *is a strongly regular graph with* $v > 4$ *vertices that is not isomorphic to* mK_{k+1} *nor* $\overline{mK_{k+1}}$*. Suppose* $v \geq 2k$*.*

Let (x, y) *be a solution given by Theorem 3.1. If* $|x| = 1$ *then*

$$\theta + \tau \in \{-2, -1, 0\}.$$

Proof. Theorem 3.1 (e) is the only case we need to consider.

Let $\Delta = \alpha^2 + 4\theta\tau(\beta + 2\theta\tau)$. Solving the quadratic in (3) gives

$$x + x^{-1} = \frac{-\alpha \pm \sqrt{\Delta}}{-2\theta\tau}.$$

If $|x| = 1$ then $x + x^{-1} = x + \overline{x} \in \mathbb{R}$ and $-2 \leq x + x^{-1} \leq 2$. We assume $\Delta \geq 0$ in this proof.

If $\theta + \tau \geq 1$, then

$$\begin{aligned} \alpha + 4\theta\tau + \sqrt{\Delta} &= v(\theta + \tau + 1) + (\theta + \tau)^2 + 4\theta\tau + \sqrt{\Delta} \\ &\geq 2v + 1 + 4\theta\tau + \sqrt{\Delta} \\ &\geq 4(k + \theta\tau) + 1 + \sqrt{\Delta} \\ &> 0. \end{aligned}$$

It follows that

$$\frac{-\alpha - \sqrt{\Delta}}{-2\theta\tau} < -2.$$

Note that $\theta > 0$, so the expression

$$\left((\alpha + 4\theta\tau) - \sqrt{\Delta}\right)\left((\alpha + 4\theta\tau) + \sqrt{\Delta}\right) = -4\theta\tau(v - 4)(\theta + \tau)^2 > 0.$$

We conclude that $\alpha + 4\theta\tau - \sqrt{\Delta} > 0$ and

$$\frac{-\alpha + \sqrt{\Delta}}{-2\theta\tau} < -2.$$

Thus $x + x^{-1} < -2$ and $|x| \neq 1$.

Suppose $\theta + \tau \leq -3$. Consider $\alpha - 4\theta\tau = (\theta + \tau)(v + \theta + \tau) + v - 4\theta\tau$. Since $v + \theta + \tau = v + a - c > 0$, we have

$$\alpha - 4\theta\tau - \sqrt{\Delta} \leq -3(v + \theta + \tau) + v - 4\theta\tau.$$

Substituting $\theta + \tau = a - c$ and $\theta\tau = c - k$ gives

$$\alpha - 4\theta\tau - \sqrt{\Delta} \leq -2(v - 2k) - c - 3a < 0.$$

It follows that

$$\frac{-\alpha + \sqrt{\Delta}}{-2\theta\tau} > 2.$$

Now the expression

$$\left((\alpha - 4\theta\tau) - \sqrt{\Delta}\right)\left((\alpha - 4\theta\tau) + \sqrt{\Delta}\right) = -4\theta\tau v(\theta + \tau + 2)^2 > 0.$$

We see that $\alpha - 4\theta\tau + \sqrt{\Delta} < 0$, so

$$\frac{-\alpha - \sqrt{\Delta}}{-2\theta\tau} > 2.$$

Therefore $x + x^{-1} > 2$ and $|x| \neq 1$. □

3.3 Theorem. *Let X be a strongly regular graph with $v > 4$ vertices, valency k, and eigenvalues k, θ and τ where $\theta \geq 0 > \tau$. Let*

$$W = I + xA(X) + yA(\overline{X})$$

where (x, y) is a solution from Theorem 3.1.

The W is a complex Hadamard matrix if and only if X or $\overline{X}$ has one of the following parameters:

i $(4\theta^2, 2\theta^2 - \theta, \theta^2 - \theta, \theta^2 - \theta)$

ii $(4\theta^2, 2\theta^2 + \theta, \theta^2 + \theta, \theta^2 + \theta)$

iii $(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2)$

iv $(4\theta^2 + 4\theta + 1, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta)$.

v $(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2)$

Proof. Observe that $\theta + \tau = 0$ if and only if $(-1 - \tau) + (-1 - \theta) = -2$. By Lemma 3.2, it is sufficient to consider only the strongly regular graphs with $\theta + \tau = 0$ or $\theta + \tau = -1$.

When $\theta + \tau = 0$, the quadratic in (3) gives

$$x + x^{-1} = -2 \quad \text{or} \quad x + x^{-1} = \frac{(-v + 2\theta^2)}{\theta^2}.$$

The former gives $x = -1$, and Theorem 3.1 (d) gives

$$y = \frac{-v + 2\theta^2 + 2 \pm \sqrt{(v-4)(v-4\theta^2)}}{2(1-\theta^2)}.$$

Note that when $v < 4\theta^2$, $y \notin \mathbb{R}$ and $|y| = 1$.

Setting $y = 1$, we get $(\theta^2 - 1)(v - 4\theta^2) = 0$. If $\theta = -\tau = 1$ then $v = k+1$ and X is complete. We conclude that $y = 1$ if and only if $v = 4\theta^2$. On the other hand, setting $y = -1$ gives $(v-4)(1-\theta^2) = 0$ which never occurs. We see that $|y| = 1$ if and only if $v \leq 4\theta^2$.

When

$$x + x^{-1} = \frac{-v + 2\theta^2}{\theta^2}$$

which is real, we have $|x + x^{-1}| \leq 2$ if and only if $v \leq 4\theta^2$. Thus $|x| = 1$ when $v \leq 4\theta^2$.

Simplifying

$$v = \frac{(k-\theta)(k+\theta)}{k-\theta^2} \leq 4\theta^2$$

gives $[k - (2\theta^2 - \theta)][k - (2\theta^2 + \theta)] \leq 0$. Then we have

$$2\theta^2 - \theta \leq k \leq 2\theta^2 + \theta.$$

The integrality of $m_\theta = \frac{1}{2}(v - 1 - \frac{k}{\theta})$ implies $\theta | k$. Hence k can be $2\theta^2 - \theta$, $2\theta^2$ or $2\theta^2 + \theta$, which give parameter set in (i), (iii) and (ii), respectively. Now (i) and (ii) are parameter sets for Latin-square graphs $L_\theta(2\theta)$ and negative Latin-square graphs $NL_\theta(2\theta)$, respectively. As expected from [8], Theomem (3.1) gives only Hadamard matrices. For parameter set (iii), we get

$$x = -1 \quad \text{and} \quad y = \frac{2\theta^2 - 3 \pm \sqrt{4\theta^2 - 5}i}{2(\theta^2 - 1)}, \tag{4}$$

or

$$x = \frac{-2\theta^2 + 1 \pm \sqrt{4\theta^2 - 1}i}{2\theta^2}$$

and by Theorem 3.1 (e)

$$y = \frac{1}{x - x^{-1}} \left(\frac{\theta\tau x - 1}{(1+\theta)(1+\tau)}(x + x^{-1} - 2 + v) - (v-2)x - 2 \right) = 1.$$

In all cases, $|x| = |y| = 1$.

We now consider $\theta + \tau = -1$. If X is a conference graph then we get the parameter set (iv) and

$$x = y^{-1} = \frac{1 \pm \sqrt{(2\theta+1)(2\theta+3)}i}{2(\theta+1)} \tag{5}$$

or

$$x = y^{-1} = \frac{-1 \pm \sqrt{(2\theta+1)(2\theta-1)}i}{2\theta}. \tag{6}$$

It follows that $|x| = |y| = 1$.

We assume θ is a positive integer. Theorem 3.1 (e) gives

$$x + x^{-1} = \frac{-1 \pm \sqrt{(2\theta+1)^4 - 4v\theta(\theta+1)}}{2\theta(\theta+1)}$$

which is real if and only if

$$v \leq \frac{(2\theta+1)^4}{4\theta(\theta+1)} = 4\theta^2+4\theta+2+\frac{1}{4\theta(\theta+1)}$$

We see that

$$v = \frac{(k-\theta)(k+\theta+1)}{k-\theta(\theta+1)} \leq 4\theta^2+4\theta+2.$$

After simplification, this inequality becomes $(k-2\theta^2-\theta)(k-2\theta^2-3\theta-1) \leq 0$. Hence we have $2\theta^2+\theta \leq k \leq 2\theta^2+3\theta+1$.

Now write $k = 2\theta^2+\theta+h$ for some integer $0 \leq h \leq 2\theta+1$. Then

$$v = \frac{(2\theta^2+h)(2\theta^2+2\theta+1+h)}{\theta^2+h} = 4\theta^2+4\theta+2+\frac{h(h-2\theta-1)}{\theta^2+h}.$$

For $v \in \mathbb{Z}$, either $h=0$, $h=2\theta+1$, or $(\theta^2+h)|(h(h-2\theta-1))$. If $h(h-2\theta-1) \neq 0$ then the last condition implies $|\theta^2+h| \leq |h(h-2\theta-1)|$. Now the inequality

$$|\theta^2+h| - |h(h-2\theta-1)| = (\theta^2+h)+h(h-2\theta-1) = (\theta-h)^2 \leq 0$$

holds if and only if $h=\theta$.

When $h=\theta$ and $k=2\theta^2+2\theta$, we get parameter set (iv) and X is a conference graph. When $h=2\theta+1$, $\overline{X}$ has parameter set (v). When $h=0$, X has parameter set (v). In this case, (3) gives $x+x^{-1}=0$ or $-\theta^{-1}(1+\theta)^{-1}$. We have

$$x = \pm i \quad \text{and} \quad y = x^{-1},$$

or

$$x = \frac{-1 \pm \sqrt{4\theta^2(\theta+1)^2-1}\; i}{2\theta(\theta+1)} \quad \text{and} \quad y = x^{-1} = \frac{-1 \mp \sqrt{4\theta^2(\theta+1)^2-1}\; i}{2\theta(\theta+1)}$$

In all cases, $|x|=|y|=1$. □

We remark on the parameter sets that give complex Hadamard matrices with non-real entries.

1. Let X be a strongly regular graph with parameters $(4\theta^2-1, 2\theta^2, \theta^2, \theta^2)$. The symplectic graph $Sp(2r)$, for $r \geq 1$, is an infinity family of graphs with these parameters [6]. Graphs of these parameters also arise from Steiner system $S(2, \theta, 2\theta^2-\theta)$.

Let $S(X) = J - I - 2A(X)$ be the Seidel matrix of X. Then

$$T = \begin{pmatrix} 0 & \mathbf{1^T} \\ \mathbf{1} & S(X) \end{pmatrix}$$

is a regular two-graph with minimal polynomial $z^2 + 2z - (4\theta^2 - 1)$. It is a generalized conference matrix with $\beta = -2$ and $T + I$ is a Hadamard matrix. The solution in (4) is the same as the one given by Theorem 4.1 of [15].

We see in Proposition 5.2 that the complex Hadamard matrices obtained from these graphs, for $\theta \geq 2$, are not of Diţă-type.

2. Let X be a strongly regular graph with parameters

$$(4\theta^2 + 4\theta + 1, 2\theta^2 + 2\theta, \theta^2 + \theta - 1, \theta^2 + \theta).$$

The Paley graphs have parameters of this form. Graphs of these parameters also arise from orthogonal arrays $OA(\theta + 1, 2\theta + 1)$.

Let $S(X)$ be the Seidel matrix of X. Then

$$T = \begin{pmatrix} 0 & \mathbf{1^T} \\ \mathbf{1} & S(X) \end{pmatrix}$$

is a regular two-graph with minimal polynomial $z^2 - (2\theta + 1)^2$. Thus T is a conference matrix and the solutions in (5) and (6) are the same as the one given by Theorem 3.1 of [15].

When $\theta^2 + \theta > 2$, $\mathcal{N}_W$ is trivial [3]. By Proposition 2.3 the complex Hadamard matrices obtained from these graphs are not of Diţă-type.

3. Let X be a strongly regular graph with parameter set

$$(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2).$$

A construction of these graph when $2\theta + 1$ is prime power can be found in [4]. Graphs of these parameters also arise from Steiner system $S(2, \theta + 1, 2\theta^2 + 2\theta + 1)$.

The Seidel matrix of X is a regular two-graph, hence a generalized conference matrix with $\beta = 0$. The complex Hadamard matrices with $x = y^{-1} = \pm i$ were constructed by Turyn in [17]. From Proposition 2.5, these complex Hadamard matrices are not of Diţă-type.

When

$$x = \frac{-1 \pm \sqrt{4\theta^2(\theta+1)^2 - 1}\; i}{2\theta(\theta+1)} \quad \text{and} \quad y = x^{-1} = \frac{-1 \mp \sqrt{4\theta^2(\theta+1)^2 - 1}\; i}{2\theta(\theta+1)},$$

we see in Proposition 5.3 that $I + xA(X) + yA(\overline{X})$ is not of Diţă-type.

4 Covers of Complete Graphs

A graph X of diameter d is *antipodal* if the relation "at distance 0 or d" is an equivalence relation. We are interested in the distance regular graphs with diameter three that are antipodal. These are the distance regular covers of complete graphs. There are three parameters (n, r, c_2) associated with each of these graphs where n is the number of equivalence classes; r is the size of each equivalence classes; and every pair of vertices at distance two have c_2 common neighbours. The intersection numbers of X depend on n, r and c_2, in particular, $a_1 = n - 2 - (r-1)c_2$.

Let X be an antipodal distance-regular graph of diameter three of parameters (n, r, c_2). The adjacency matrix of X has four eigenvalues:

$$n-1, \quad -1, \quad \theta = \frac{\delta + \sqrt{\delta^2 + 4(n-1)}}{2}, \quad \text{and} \quad \tau = \frac{\delta - \sqrt{\delta^2 + 4(n-1)}}{2},$$

where $\delta = n - 2 - rc_2$.

4.1 Lemma. *[7] If $\delta = 0$ then $\theta = -\tau = \sqrt{n-1}$. Otherwise, θ and τ are integers.* □

From Theorem 9.1 and its proof in [2], we see that

4.2 Lemma. *[2] For $i = 1, 2, 3$, let A_i be the i-th distance matrix of an antipodal distance regular graph of diameter three with parameters (n, r, c_2). Then*

$$W = I + xA_1 + yA_2 + zA_3$$

is type II if and only if the following system of equations have a solution:

$$(1 - x - (r-1)y + (r-1)z)\left(1 - \frac{1}{x} - \frac{(r-1)}{y} + \frac{(r-1)}{z}\right) = nr, \qquad (7)$$

$$(1 + \theta x - \theta y - z)(1 + \frac{\theta}{x} - \frac{\theta}{y} - \frac{1}{z}) = nr, \qquad (8)$$

$$(1 + \tau x - \tau y - z)(1 + \frac{\tau}{x} - \frac{\tau}{y} - \frac{1}{z}) = nr. \qquad (9)$$

□

To get complex Hadamard matrices, we add the condition $|x| = |y| = |z| = 1$.

4.3 Lemma. *If the system of equations in Lemma 4.2 has a solution satisfying $|x| = |y| = |z| = 1$ then $n \leq 16$.*

Proof. Assume $|x| = |y| = |z| = 1$. Then for $\alpha \in \{x, y, z, \frac{x}{y}, \frac{x}{z}, \frac{y}{z}\}$, we have $\alpha + \alpha^{-1} \in \mathbb{R}$ and $|\alpha + \alpha^{-1}| \leq 2$.

Expanding the right-hand side of Equation (7) gives

$$\begin{aligned} nr &= 2 + 2(r-1)^2 - (x + \frac{1}{x}) - (r-1)(y + \frac{1}{y}) + (r-1)(z + \frac{1}{z}) \\ &\quad + (r-1)(\frac{y}{x} + \frac{x}{y}) - (r-1)(\frac{x}{z} + \frac{z}{x}) - (r-1)^2(\frac{z}{y} + \frac{y}{z}) \\ &\leq 4r^2. \end{aligned} \tag{10}$$

Similarly, Equations (8) and (9) yield

$$\begin{aligned} nr &= 2 + 2\theta^2 + \theta(x + \frac{1}{x}) - \theta(y + \frac{1}{y}) - (z + \frac{1}{z}) - \theta^2(\frac{y}{x} + \frac{x}{y}) - \theta(\frac{z}{x} + \frac{x}{z}) + \theta(\frac{z}{y} + \frac{y}{z}) \\ &\leq 4(1+\theta)^2 \end{aligned} \tag{11}$$

and

$$\begin{aligned} nr &= 2 + 2\tau^2 + \tau(x + \frac{1}{x}) - \tau(y + \frac{1}{y}) - (z + \frac{1}{z}) - \tau^2(\frac{y}{x} + \frac{x}{y}) - \tau(\frac{z}{x} + \frac{x}{z}) + \tau(\frac{z}{y} + \frac{y}{z}) \\ &\leq 2 + 2\tau^2 - 2\tau - 2\tau + 2 + 2\tau^2 - 2\tau - 2\tau \\ &= 4(1-\tau)^2, \end{aligned} \tag{12}$$

respectively.

When $\delta \geq 2$, the expression

$$\begin{aligned} n^2 - 16(1-\tau)^2 &= n^2 - 4(\delta-2)^2 - 4(\delta^2 + 4(n-1)) + 8(\delta-2)\sqrt{\delta^2 + 4(n-1)} \\ &\geq n^2 - 4(\delta-2)^2 - 4(\delta^2 + 4(n-1)) + 8(\delta-2)\delta \\ &= n(n-16). \end{aligned}$$

If $n \geq 17$ then $n^2 > 16(1-\tau)^2$, it follows from (12) that $n > 4r$.

When $\delta = 1$, we have $\tau = \frac{1-\sqrt{4n-3}}{2}$ and

$$nr \leq 4(1-\tau)^2 = 4n - 2 + 2\sqrt{4n-3} < 4n + 4\sqrt{n}.$$

When $n \geq 17$, $r \leq \lfloor 4 + \frac{4}{\sqrt{n}} \rfloor = 4$ and (10) does not hold.

When $\delta = 0$, $\theta = -\tau = \sqrt{n-1}$. Adding both sides of Equations (8) and (9) gives

$$2nr = 4n - 2(z + \frac{1}{z}) - 2(n-1)(\frac{y}{x} + \frac{x}{y}) \leq 8n.$$

Hence $r \leq 4$ and (10) does not hold for $n \geq 17$.

When $\delta = -1$, we have $\theta = \frac{-1+\sqrt{4n-3}}{2}$ and

$$nr \leq 4(1+\theta)^2 = 4n - 2 + 2\sqrt{4n-3}.$$

Using the same argument for $\delta = 1$, we see that if $n \geq 17$ then $nr \leq 4(1+\theta)^2$ implies $n > 4r$.

When $\delta \leq -2$, the expression

$$\begin{aligned} n^2 - 16(1+\theta)^2 &= n^2 - 8\delta^2 - 16\delta - 16n - 8(\delta+2)\sqrt{\delta^2 + 4(n-1)} \\ &\geq n^2 - 16n - 8\delta^2 - 16\delta - 8(\delta+2)(-\delta) \\ &= n(n-16). \end{aligned}$$

If $n \geq 17$ then $n^2 > 16(1+\theta)^2$, but (11) gives $n > 4r$.

We conclude that when $n \geq 17$, (10), (11) and (12) do not hold simultaneously. □

4.4 Theorem. *There are only finitely many antipodal distance-regular graphs of diameter three where the span of $\{I, A_1, A_2, A_3\}$ contains a complex Hadamard matrix.*

Proof. Since $a_1 \geq 0$, we have $(r-1)c_2 \leq n-2$. Therefore, there are only finitely many parameter sets (n, r, c_2) where $n \leq 16$. □

Computations in Maple revealed only three graphs that give complex Hadamard matrices:

1. The cycle on six vertices: its parameters are $(3, 2, 1)$ and

   $$y = \frac{-1 \pm \sqrt{3}\, i}{2}, \quad z = \pm i, \quad x = yz.$$

   We get the matrix F_6 in [16].

2. The cube: its parameters are $(4, 2, 2)$ and $x = \pm i$, $y = -1$ and $z = -x$. We get the complex Hadamard matrix

   $$\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{\otimes 3}$$

3. The line graph of the Petersen graph: its parameters are $(5, 3, 1)$ and

$$\begin{aligned}
&x = 1, && y = \frac{-7 \pm \sqrt{15}\, i}{8}, && z = 1; \\
&x = \frac{5 \pm \sqrt{11}\, i}{6}, && y = -1, && z = x; \quad \text{or} \\
&x = \frac{-1 \pm \sqrt{15}\, i}{4}, && y = x^{-1}, && z = 1.
\end{aligned}$$

Computation in Maple showed that $\mathcal{N}_W$ is trivial for all (x, y) given here. By Proposition 2.3, none of these matrices is of Diţă-type.

5 Computation

Let X be a strongly regular graph that has parameters in Cases (c) and (e) of Theorem 3.1. We show that the complex Hadamard matrix in the adjacency algebra of X is not of Diţă-type.

We adapt the computation in [3] to these strongly regular graphs. The following is Lemma 3.1 of [3] which was phrased for conference graphs but the statement applies to all strongly regular graphs.

5.1 Lemma. *Let X be a strongly regular graph on $v \geq 5$ vertices with degree $k \geq 2$. Let $W = I + xA(X) + yA(\overline{X})$ be a complex Hadamard matrix. If the Hermitian product $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is non-zero for all adjacent vertices α and β and for all $\gamma \neq \alpha, \beta$, then $\mathcal{N}_W$ is trivial.* □

Let α, β and γ be vertices of a strongly regular graph X with parameters (v, k, a, c). For $x_\alpha \in \{\alpha, \bar{\alpha}\}$, $x_\beta \in \{\beta, \bar{\beta}\}$ and $x_\gamma \in \{\gamma, \bar{\gamma}\}$, we define $\Gamma_{x_\alpha x_\beta x_\gamma}$ to be the set of vertices that are adjacent (not adjacent) to x_v if $x_v = v$ ($x_v = \bar{v}$, respectively), for $v = \alpha, \beta, \gamma$. For instance $\Gamma_{\alpha\beta\gamma}$ is the set of common neighbours of α, β and γ while $\Gamma_{\alpha\beta\bar{\gamma}}$ is the set of common neighbours of α and β that are not adjacent to γ. Now the vertex set of X is partitioned into

$$\{\alpha, \beta, \gamma\} \cup \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}.$$

Let $W = I + xA(X) + yA(\overline{X})$ be a complex Hadamard matrix. From (2), we have

$$Y_{\alpha,\beta}(u)\overline{Y_{\gamma,\alpha}(u)} = Y_{\alpha,\beta}(u)Y_{\alpha,\gamma}(u) = \frac{W(u,\alpha)^2}{W(u,\beta)W(u,\gamma)}.$$

It is easy to verify that

$$Y_{\alpha,\beta}(u)\overline{Y_{\gamma,\alpha}(u)} = \begin{cases} 1 & \text{if } u \in \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \\ xy^{-1} & \text{if } u \in \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\gamma} \\ x^{-1}y & \text{if } u \in \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \\ x^{-2}y^2 & \text{if } u \in \Gamma_{\bar{\alpha}\beta\gamma} \\ x^2y^{-2} & \text{if } u \in \Gamma_{\alpha\bar{\beta}\bar{\gamma}}. \end{cases}$$

Hence the Hermitian product

$$\begin{aligned} < Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \quad = \quad & \frac{1}{W(\alpha,\beta)W(\alpha,\gamma)} + \frac{W(\beta,\alpha)^2}{W(\beta,\gamma)} + \\ & \frac{W(\gamma,\alpha)^2}{W(\gamma,\beta)} + |\Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}}|xy^{-1} + \\ & |\Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma}|x^{-1}y + |\Gamma_{\bar{\alpha}\beta\gamma}|x^{-2}y^2 + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}|x^2y^{-2}. \end{aligned} \tag{13}$$

In the following computation, we let α and β be adjacent vertices in X.

1. We first consider the case where γ is adjacent to both α and β. Then $W(\alpha,\beta) = W(\alpha,\gamma) = W(\beta,\gamma) = x$. We use Γ_v to denote the set of neighbours of v in X. Then we get

$$\begin{aligned} \Gamma_\alpha &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \cup \{\beta,\gamma\} \\ \Gamma_\beta &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \{\alpha,\gamma\} \\ \Gamma_\gamma &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \cup \{\alpha,\beta\} \\ \Gamma_\alpha \cap \Gamma_\beta &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \{\gamma\} \\ \Gamma_\alpha \cap \Gamma_\gamma &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \{\beta\} \\ \Gamma_\beta \cap \Gamma_\gamma &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \{\alpha\} \\ V(X) &= \Gamma_{\alpha\beta\gamma} \cup \Gamma_{\bar{\alpha}\beta\gamma} \cup \Gamma_{\alpha\bar{\beta}\gamma} \cup \Gamma_{\alpha\beta\bar{\gamma}} \cup \Gamma_{\alpha\bar{\beta}\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\beta\bar{\gamma}} \cup \Gamma_{\bar{\alpha}\bar{\beta}\gamma} \cup \Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \cup \{\alpha,\beta,\gamma\}. \end{aligned}$$

   Now we translate the above to the following system of equations.

$$\begin{aligned} k &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| + 2 \\ k &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| + 2 \\ k &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| + 2 \\ a &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + 1 \\ a &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + 1 \\ a &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + 1 \\ v &= |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| + |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| + 3. \end{aligned}$$

Solving this system of equations, we get

$$\begin{array}{rcl} |\Gamma_{\alpha\beta\gamma}| &=& m, \\ |\Gamma_{\bar{\alpha}\beta\gamma}| &=& |\Gamma_{\alpha\bar{\beta}\gamma}| = |\Gamma_{\alpha\beta\bar{\gamma}}| = a-1-m, \\ |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| &=& |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| = |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| = k-2a+m, \quad \text{and} \\ |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| &=& v-3k+3a-m, \end{array}$$

for some integer m. Using Equation (13), we have

$$\begin{array}{rl} & < Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \qquad (14) \\ = & x^{-2} + 2x + (v-3k+3a) + 2(a-1-m)xy^{-1} + 2(k-2a+m)x^{-1}y \\ & +(a-1-m)x^{-2}y^2 + (k-2a+m)x^2y^{-2}. \end{array}$$

2. Suppose γ is adjacent to α but not to β which gives

$$W(\alpha,\beta) = W(\alpha,\gamma) = x \quad \text{and} \quad W(\beta,\gamma) = y.$$

We get

$$\begin{array}{l} k = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| + 2 \\ k = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| + 1 \\ k = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| + 1 \\ a = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| \\ a = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| \\ c = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + 1 \\ v = |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar{\alpha}\beta\gamma}| + |\Gamma_{\alpha\bar{\beta}\gamma}| + |\Gamma_{\alpha\beta\bar{\gamma}}| + |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| + |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| + |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| + 3. \end{array}$$

Solving this system of equations yields

$$\begin{array}{rcl} |\Gamma_{\alpha\beta\gamma}| &=& m, \\ |\Gamma_{\bar{\alpha}\beta\gamma}| &=& c-1-m \\ |\Gamma_{\alpha\bar{\beta}\gamma}| &=& |\Gamma_{\alpha\beta\bar{\gamma}}| = a-m, \\ |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| &=& k-2a-2+m \\ |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| &=& |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| = k-a-c+m, \quad \text{and} \\ |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| &=& v-3k+2a+c-m, \end{array}$$

for some integer m. Using Equation (13), we have

$$\begin{array}{rl} & < Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \qquad (15) \\ = & x^{-2} + 2x^2y^{-1} + (v-3k+2a+c) + 2(a-m)xy^{-1} \\ & +2(k-a-c+m)x^{-1}y + (c-1-m)x^{-2}y^2 + (k-2a-2+m)x^2y^{-2}. \end{array}$$

3. Suppose γ is adjacent to β but not to α which gives

$$W(\alpha,\beta)=W(\beta,\gamma)=x \quad \text{and} \quad W(\alpha,\gamma)=y.$$

We get

$$\begin{aligned}
k =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}| + |\Gamma_{\alpha\bar\beta\bar\gamma}| + 1\\
k =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}| + |\Gamma_{\bar\alpha\beta\bar\gamma}| + 2\\
k =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + |\Gamma_{\bar\alpha\bar\beta\gamma}| + 1\\
a =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}|\\
c =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + 1\\
a =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}|\\
v =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}| + |\Gamma_{\alpha\bar\beta\bar\gamma}| + |\Gamma_{\bar\alpha\beta\bar\gamma}| + |\Gamma_{\bar\alpha\bar\beta\gamma}| + |\Gamma_{\bar\alpha\bar\beta\bar\gamma}| + 3.
\end{aligned}$$

Solving this system of equations gives

$$\begin{aligned}
|\Gamma_{\alpha\beta\gamma}| &= m,\\
|\Gamma_{\bar\alpha\beta\gamma}| &= |\Gamma_{\alpha\beta\bar\gamma}| = a-m,\\
|\Gamma_{\alpha\bar\beta\gamma}| &= c-1-m,\\
|\Gamma_{\alpha\bar\beta\bar\gamma}| &= |\Gamma_{\bar\alpha\bar\beta\gamma}| = k-a-c+m,\\
|\Gamma_{\bar\alpha\beta\bar\gamma}| &= k-2a-2+m, \quad \text{and}\\
|\Gamma_{\bar\alpha\bar\beta\bar\gamma}| &= v-3k+2a+c-m,
\end{aligned}$$

for some integer m. Using Equation (13), we have

$$\begin{aligned}
& < Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \qquad (16)\\
=\ & x^{-1}y^{-1} + x + x^{-1}y^2 + (v-3k+2a+c) + (a+c-1-2m)xy^{-1}\\
& +(2k-3a-c-2+2m)x^{-1}y + (a-m)x^{-2}y^2 + (k-a-c+m)x^2y^{-2}.
\end{aligned}$$

4. Suppose γ is not adjacent to α nor to β which gives $W(\alpha,\beta)=x$ and $W(\alpha,\gamma)=W(\beta,\gamma)=y$. We get

$$\begin{aligned}
k =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}| + |\Gamma_{\alpha\bar\beta\bar\gamma}| + 1\\
k =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}| + |\Gamma_{\bar\alpha\beta\bar\gamma}| + 1\\
k =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + |\Gamma_{\bar\alpha\bar\beta\gamma}|\\
a =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}|\\
c =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}|\\
c =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}|\\
v =& |\Gamma_{\alpha\beta\gamma}| + |\Gamma_{\bar\alpha\beta\gamma}| + |\Gamma_{\alpha\bar\beta\gamma}| + |\Gamma_{\alpha\beta\bar\gamma}| + |\Gamma_{\alpha\bar\beta\bar\gamma}| + |\Gamma_{\bar\alpha\beta\bar\gamma}| + |\Gamma_{\bar\alpha\bar\beta\gamma}| + |\Gamma_{\bar\alpha\bar\beta\bar\gamma}| + 3.
\end{aligned}$$

Solving this system of equations, we get

$$\begin{array}{rcl} |\Gamma_{\alpha\beta\gamma}| & = & m, \\ |\Gamma_{\bar{\alpha}\beta\gamma}| & = & |\Gamma_{\alpha\bar{\beta}\gamma}| = c - m, \\ |\Gamma_{\alpha\beta\bar{\gamma}}| & = & a - m, \\ |\Gamma_{\alpha\bar{\beta}\bar{\gamma}}| & = & |\Gamma_{\bar{\alpha}\beta\bar{\gamma}}| = k - a - c - 1 + m, \\ |\Gamma_{\bar{\alpha}\bar{\beta}\gamma}| & = & k - 2c + m, \quad \text{and} \\ |\Gamma_{\bar{\alpha}\bar{\beta}\bar{\gamma}}| & = & v - 3k + a + 2c - 1 - m, \end{array}$$

for some integer m. Using Equation (13), we have

$$\begin{array}{rl} & < Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \\ = & x^{-1}y^{-1} + y + x^2y^{-1} + (v - 3k + 2a + c - 1) + (a + c - 2m)xy^{-1} \\ & +(2k - a - 3c - 1 + 2m)x^{-1}y + (c - m)x^{-2}y^2 + (k - a - c - 1 + m)x^2y^{-2}. \end{array} \tag{17}$$

5.2 Proposition. *Let X be a strongly regular graph with parameters*

$$(4\theta^2 - 1, 2\theta^2, \theta^2, \theta^2), \quad \textit{for } \theta \geq 2.$$

Let $W = I + xA(X) + yA(\overline{X})$ where

$$x = -1 \quad \text{and} \quad y = \frac{2\theta^2 - 3 \pm \sqrt{4\theta^2 - 5}\; i}{2(\theta^2 - 1)}$$

or

$$x = \frac{-2\theta^2 + 1 \pm \sqrt{4\theta^2 - 1}\; i}{2\theta^2} \quad \textit{and} \quad y = 1.$$

Then $\mathcal{N}_W$ is trivial. Consequently, W is not of Diţă-type.

Proof. Suppose

$$x = -1 \quad \text{and} \quad y = \frac{2\theta^2 - 3 \pm \sqrt{4\theta^2 - 5}\; i}{2(\theta^2 - 1)}.$$

Using Equation (15), the real part of $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is

$$\frac{-(4\theta^2 - 5)(\theta^2 - 3)}{2(\theta^2 - 1)^2}.$$

Using Equations (14), (16) and (17), the real part of $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is

$$\frac{-(4\theta^2 - 5)}{2(\theta^2 - 1)}$$

for all the other vertex γ in $V(X)\backslash\{\alpha,\beta\}$. If $\theta \geq 2$ then $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \neq 0$ for all $\gamma \neq \alpha, \beta$. By Lemma 5.1, $\mathcal{N}_W$ is trivial.

Suppose

$$x = \frac{-2\theta^2 + 1 \pm \sqrt{4\theta^2 - 1}\; i}{2\theta^2} \quad \text{and} \quad y = 1.$$

Using Equations (14), (15), (16) and (17), the real part of $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is

$$\frac{-(4\theta^2 - 1)}{2\theta^2}$$

which is non-zero for $\gamma \neq \alpha, \beta$ and for $\theta \geq 1$. By Lemma 5.1, $\mathcal{N}_W$ is trivial.
□

5.3 Proposition. *Let X be a strongly regular graph with parameters*

$$(4\theta^2 + 4\theta + 2, 2\theta^2 + \theta, \theta^2 - 1, \theta^2).$$

Let $W = I + xA(X) + yA(\overline{X})$ where

$$x = \frac{-1 \pm \sqrt{4\theta^2(\theta+1)^2 - 1}\; i}{2\theta(\theta+1)} \quad \text{and} \quad y = x^{-1} = \frac{-1 \mp \sqrt{4\theta^2(\theta+1)^2 - 1}\; i}{2\theta(\theta+1)}$$

Then $\mathcal{N}_W$ is trivial. Consequently, W is not of Diţă-type.

Proof. If γ is adjacent to both α and β then, by Equation (14), the real part of $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is

$$\frac{-(\theta^2 + \theta + 1)(2\theta^2 + 2\theta + 1)(2\theta^2 + 2\theta - 1)}{2\theta^3(1+\theta)^3}.$$

If γ is adjacent to α but not to β then by Equation (15), the real part of $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is

$$\frac{-(2\theta^2 + 2\theta - 1)(2\theta^2 + 2\theta + 1)(\theta^4 + 2\theta^3 - \theta - 1)}{2\theta^4(1+\theta)^4}.$$

If γ is not adjacent to α then by Equations (16) and (17), the real part of $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} >$ is

$$\frac{(2\theta^2 + 2\theta - 1)(2\theta^2 + 2\theta + 1)}{2\theta^2(1+\theta)^2}.$$

We have $< Y_{\alpha,\beta}, Y_{\gamma,\alpha} > \neq 0$, for $\gamma \neq \alpha, \beta$ and for $\theta \geq 1$. By Lemma 5.1, $\mathcal{N}_W$ is trivial. □

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