I. Electrostatics

Electric charge: basic tenets

(1) particles have fundamental property called electric charge (like mass).

			mass	charge
make up atoms	proton	p	1.673×10^{-27} kg	$+1.602 \times 10^{-19}$ C $= +e$
	neutron	n	1.675×10^{-27} kg	0
	electron	e^-	9.109×10^{-31} kg	-1.602×10^{-19} C $= -e$

C = coulomb = SI unit of electric charge
$= 6.241 \times 10^{18}\, e$ (same charge as 6.241×10^{18} protons)

(2) electric charge is additive (like mass).
Say you have two particles with masses m_1 & m_2 and charges q_1 & q_2:

Total mass $= m_1 + m_2$
Total charge $= q_1 + q_2$

(3) Electric charge can be either positive or negative (unlike mass, which is always positive)

(4) Opposite charges attract, like charges repel. (Unlike mass — gravity is always attractive)

(5) charge is <u>conserved</u> (cannot be created or destroyed).

But charge can be exchanged in elementary particle reactions like radioactive decay.

e.g. $^{40}K \longrightarrow {}^{40}Ar + \underbrace{\nu \text{ (neutrino) } \{+ \gamma \text{ (photon)}\}}_{\text{both have zero charge}}$

$$p + e^- \longrightarrow n + \nu$$

Net charge: $+e - e \longrightarrow 0e + 0e$

(6) Charge is <u>quantized</u>. Elementary particles only come with certain allowed charges (in units of e)

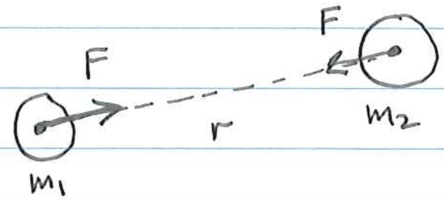
electron : $q = -e$
quarks : $q = +\frac{2}{3}e$ or $-\frac{1}{3}e$

proton = 3 quarks $\Rightarrow$ $q = +\frac{2}{3}e + \frac{2}{3}e - \frac{1}{3}e = +e$
neutron = 3 quarks $\Rightarrow$ $q = \frac{2}{3}e - \frac{1}{3}e - \frac{1}{3}e = 0e$.

Why does e^- and p have <u>exactly</u> opposite charges?

Coulomb's law

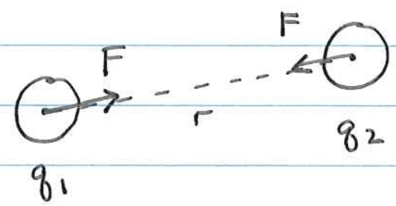
First recall Newton's law of gravitation: mass m_1 & m_2



$$F = \frac{G m_1 m_2}{r^2}$$

G = Newton's constant = strength of gravitational force.
$= 6.674 \times 10^{-11}\ N m^2 / kg^2$

Coulomb's law: Force between charges q_1 & q_2



$$F = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

$\frac{1}{4\pi\varepsilon_0}$ = constant governing strength of electric force.

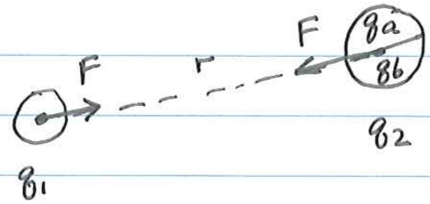
$$= 8.988 \times 10^{9}\ \frac{N\,m^2}{C^2}$$

ε_0 = "permittivity of free space"
$= 8.854 \times 10^{-12}\ C^2/Nm^2$

Why does Coulomb's law have the form it does?

$F \propto q_1 q_2$ because charge is additive.

Suppose charge 2 is divided into q_a and q_b such that $q_2 = q_a + q_b$. Consider the separate forces acting between "1" and "a", and "1" and "b"

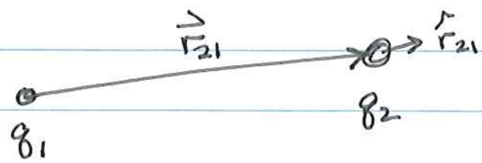


$$F = F_a + F_b = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_a}{r^2} + \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_b}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{q_1(q_a+q_b)}{r^2}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

Superposition principle: net force = sum of individual forces.

Vector form of Coulomb's law:



$$\vec{F}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21}$$

Force acting on 2.

$\vec{r}_{21}$ = distance vector from 1 to 2 $= -\vec{r}_{12}$

$r_{21} = |\vec{r}_{21}|$ = magnitude $= |\vec{r}_{12}|$

$\hat{r}_{21} = \vec{r}_{21}/|\vec{r}_{21}|$ = unit vector $= -\hat{r}_{12}$

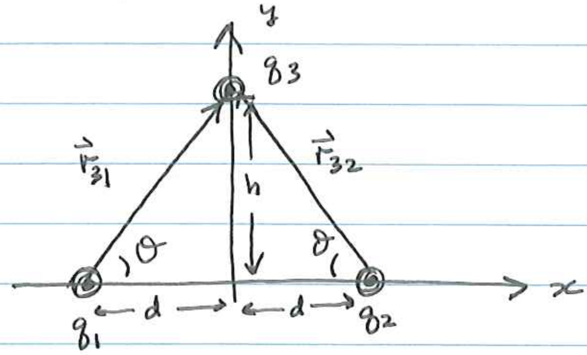
$$\vec{F}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} = -\vec{F}_2$$

Forces are equal and opposite.

Taking the magnitude:

$$|\vec{F}_1| = |\vec{F}_2| = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \qquad (\text{with } r_{12} = r_{21} = r)$$

<u>Example</u>: calculate forces in a three charge configuration.



What is the net force acting on q_3?

$$\vec{F}_3 = \vec{F}_{31} \text{ (force on 3 due to 1)} + \vec{F}_{32} \text{ (force on 3 due to 2)}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{r_{31}^2}\hat{r}_{31} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{r_{32}^2}\hat{r}_{32}$$

Don't forget that you are adding <u>vectors</u>!

$$\vec{r}_{31} = (0, h, 0) - (-d, 0, 0) = (d, h, 0)$$

$$\vec{r}_{32} = (-d, h, 0)$$

$$|\vec{r}_{31}| = |\vec{r}_{32}| = \sqrt{d^2 + h^2}$$

$$\hat{r}_{31} = \frac{1}{\sqrt{d^2+h^2}}(d, h, 0) = (\cos\theta, \sin\theta, 0)$$

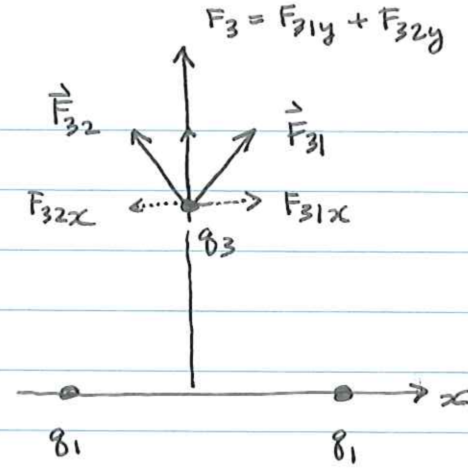
$$\hat{r}_{32} = \frac{1}{\sqrt{d^2+h^2}}(-d, h, 0) = (-\cos\theta, \sin\theta, 0)$$

$$\vec{F}_3 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_3}{d^2+h^2}\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix} + \frac{1}{4\pi\varepsilon_0}\frac{q_2 q_3}{d^2+h^2}\begin{pmatrix}-\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q_3}{d^2+h^2}\left[(q_1 - q_2)\cos\theta\,\hat{x} + (q_1 + q_2)\sin\theta\,\hat{y}\right]$$

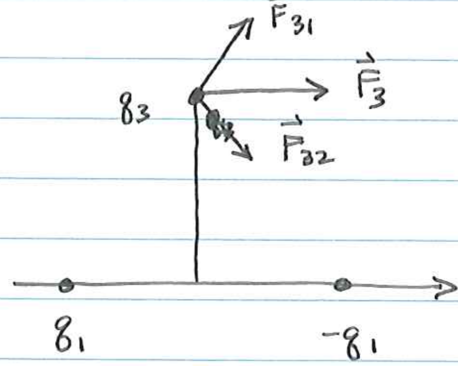
Note: $\hat{x} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\hat{y} = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\hat{z} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

case 1: equal charges $q_1 = q_2$
(arrows drawn for $q_1 q_3 > 0$ – repulsive)



x-components cancel out by symmetry.
Configuration symmetric with respect to reflection about y-axis.

case 2: opposite charges with $q_1 = -q_2$.
(electric dipole)



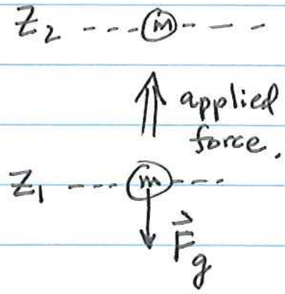
y-components cancel out

Energy and Work

Recall concept of potential energy associated with gravitational force.

grav. force: $\vec{F}_g = -mg\,\hat{z}$

Work required to move a mass m from height $z_1 \to z_2$:



$$W = (\text{applied force}) \times (\text{displacement})$$

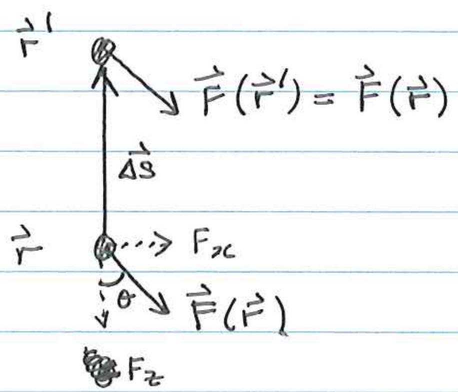
$$= -F_g \times (z_2 - z_1) = mg(z_2 - z_1)$$

Potential energy U gained by mass m: $\Delta U = mg(z_2 - z_1)$

Need more general formula for work. Previous result assumed (1) $\vec{F}$ is const (doesn't depend on position), and (2) displacement is parallel to $\vec{F}$.

Consider mass m at position $\vec{r} = (x, y, z)$, which experiences a force $\vec{F}(\vec{r})$. What is the work required to move it to a new position $\vec{r}\,' = \vec{r} + \vec{\Delta S} = (x+\Delta x, y+\Delta y, z+\Delta z)$?

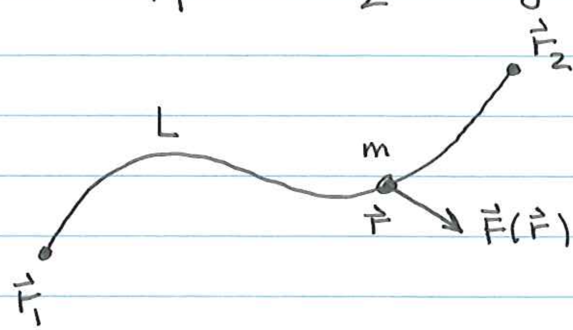
Assume displacement $\vec{\Delta S} = (\Delta x, \Delta y, \Delta z)$ is small so that $\vec{F}$ is constant from $\vec{r}$ to $\vec{r}\,'$.



Only component of $\vec{F}$ parallel to $\vec{\Delta S}$ (F_z) requires work to move m.

$$\Delta W = -F_z |\vec{\Delta S}| = - |\vec{F}| \cos\theta \, |\vec{\Delta S}| = -\vec{F}(\vec{r}) \cdot \vec{\Delta S}$$

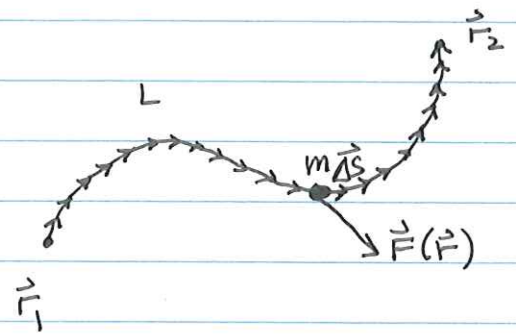
Next, consider an arbitrary force $\vec{F}(\vec{r})$. What is the work required to move m ~~along a line~~ from $\vec{r}_1$ to $\vec{r}_2$ along an arbitrary line L?



L is the path mass m takes from $\vec{r}_1$ to $\vec{r}_2$.
At each point $\vec{r}$ along the way, it experiences force $\vec{F}(\vec{r})$.

Note: total displacement $\vec{r}_2 - \vec{r}_1$ is <u>not</u> small, so $\vec{F}(\vec{r})$ is not necessarily constant along L.

Divide L into infinitesimally small segments $\vec{\Delta s}$. $\vec{F}$ is constant along each segment individually.



along each segment, the work is $\Delta W = -\vec{F}(\vec{r}) \cdot \vec{\Delta s}$

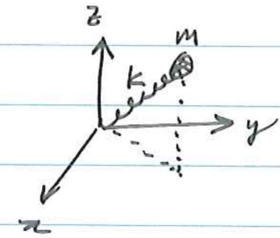
Total work = sum over all segments $= \sum \Delta W = -\sum \vec{F} \cdot \vec{\Delta s}$
Limit that $\vec{\Delta s}$ becomes infinitesimal: $\vec{\Delta s} \to \vec{ds}$

$$\text{Total work} = W = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}) \cdot \vec{ds} = -\int_L \vec{F}(\vec{r}) \cdot \vec{ds}$$

This is called a <u>line integral</u>.

What is $\vec{ds}$? It is a vector $\vec{ds} = (dx, dy, dz)$ that points along the line L.

Example: consider a force $\vec{F}(\vec{r}) = -k\vec{r}$
(spring in 3D, k = spring const)

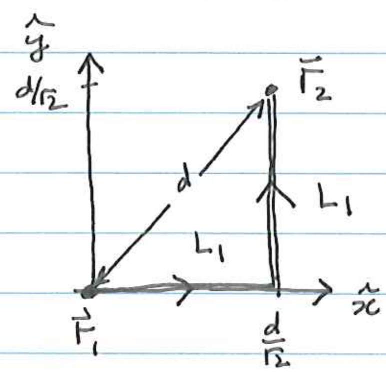


Recall: 1D $\quad F = -kx, \quad W = \frac{1}{2}k(x_2^2 - x_1^2)$
to stretch from $x_1 \to x_2$.

What is the work required to move mass from $\vec{r}_1 = 0$ to $\vec{r}_2 = \frac{1}{\sqrt{2}}(d, d, 0)$ along the following lines?

(note: total displacement is $|\vec{r}_2 - \vec{r}_1| = d$)

(1) Line L_1: first from $\vec{r} = \vec{r}_1 = 0$ to $\vec{r} = \frac{1}{\sqrt{2}}(d, 0, 0)$
then from $\vec{r} = \frac{1}{\sqrt{2}}(d, 0, 0)$ to $\vec{r} = \vec{r}_2$.



1st segment: $\vec{ds} = (dx, 0, 0)$ only x is changing while $y = z = 0$.

2nd segment: $\vec{ds} = (0, dy, 0)$ only y is changing while $z = 0$ and $x = d/\sqrt{2}$

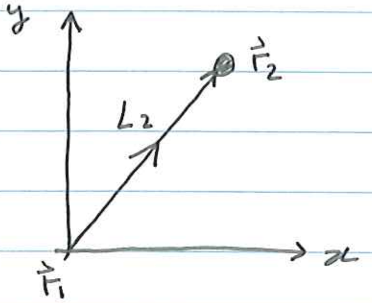
$$W = -\int_{L_1} \vec{F} \cdot \vec{ds} = -\int_0^{d/\sqrt{2}} dx\, (-k)\, (x, 0, 0) \cdot (dx, 0, 0)$$

$$- \int_0^{d/\sqrt{2}} dy\, (-k) \left(\frac{d}{\sqrt{2}}, y, 0\right) \cdot (0, dy, 0)$$

$$= k \int_0^{d/\sqrt{2}} dx\, x + k \int_0^{d/\sqrt{2}} dy\, y = k \frac{d^2}{4} \times 2 = \frac{1}{2} k d^2$$

$$W = \frac{1}{2} k \left(|\vec{r}_2|^2 - |\vec{r}_1|^2 \right)$$ usual work to stretch a spring

(2) Line L_2: diagonal line from $\vec{r}_1$ to $\vec{r}_2$.



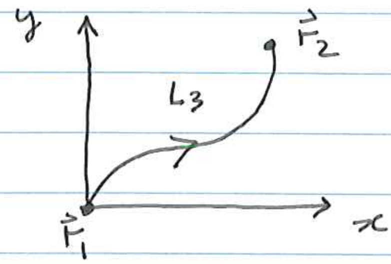
Now both x and y are changing, but along line $y = x$. $\Rightarrow$ $dy = dx$, and $z = 0$. $(dz = 0)$

$$\vec{ds} = (dx, dy, dz) = (dx, dx, 0)$$

$$W = -\int_{L_2} \vec{F} \cdot \vec{ds} = k \int_0^{d/\sqrt{2}} (x, x, 0) \cdot (dx, dx, 0)$$

$$= 2k \int_0^{d/\sqrt{2}} dx \; x = \frac{1}{2} k d^2$$

(3) Line L_3: arbitrary line described by function $y(x)$.



Again, both x and y are changing, but along line $y = y(x)$. (and $z = 0$)

Then $dy = \frac{dy}{dx} dx = y'(x)\, dx$

So $\vec{ds} = (dx, y'(x)\, dx, 0)$, $\vec{F} = -k(x, y(x), 0)$

$$W = k \int_0^{d/\sqrt{2}} (dx, y'(x)\, dx, 0) \cdot (x, y(x), 0)$$

$$= k \int_0^{d/\sqrt{2}} dx \left(x + y'(x)\, y(x) \right)$$

$\uparrow$ note: $dx\, y'\, y = dx \frac{dy}{dx} y = dy\, y$

$\cancel{= \frac{1}{2} k \left(\frac{d}{\sqrt{2}}\right)^2 + \frac{1}{2} k \left(\frac{d}{\sqrt{2}}\right)^2}$

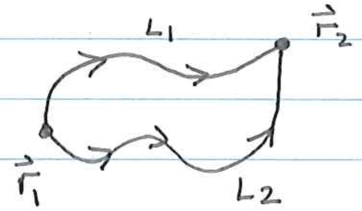
example: $y(x) = \frac{2}{d} x^3$, $y'(x) = \frac{6}{d} x^2$

$$= k \cdot \frac{1}{2} \left(\frac{d}{\sqrt{2}}\right)^2 + \frac{1}{2} k \, \underbrace{y\left(\frac{d}{\sqrt{2}}\right)^2}_{y(d/\sqrt{2}) = \frac{d}{\sqrt{2}}} = \frac{1}{2} k d^2$$

Forces may be conservative or nonconservative

- Conservative force: the work W to go from $\vec{r}_1$ to $\vec{r}_2$ is independent of path taken.

Consider two lines L_1, L_2 starting at $\vec{r}_1$ and ending at $\vec{r}_2$.



$$W_1 = -\int_{L_1} \vec{F} \cdot \vec{ds} = W(L_1)$$
$$W_2 = -\int_{L_2} \vec{F} \cdot \vec{ds} = W(L_2)$$

$W_1 = W_2$ for conservative force, for any L_1, L_2.

corollary: $W = 0$ for any path with $\vec{r}_1 = \vec{r}_2$ (closed loop)

Define a path "$-L_2$" which is same as L_2, but in opposite direction ($\vec{r}_2 \to \vec{r}_1$).

$$W(-L_2) = -\int_{\vec{r}_2}^{\vec{r}_1} \vec{ds} \cdot \vec{F} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{ds} \cdot \vec{F} = -W(L_2)$$

The total work to go from $\vec{r}_1 \to \vec{r}_2$ along L_1 and back from $\vec{r}_2 \to \vec{r}_1$ along L_2 is:

$$W = W(L_1) + W(-L_2) = W(L_1) - W(L_2) = 0$$

Since work is same along any path, you can choose the path to make the calculation easiest.

Examples of conservative forces: gravity, Coulomb, spring forces
Nonconservative forces: e.g. friction (depends on path taken)

$$W = -\int \vec{ds} \cdot \vec{F}_{21} = -\int_{\infty}^{r_{12}} dr \frac{q_1 q_2}{4\pi\varepsilon_0 r^2} = \frac{q_1 q_2}{4\pi\varepsilon_0 r} = U$$

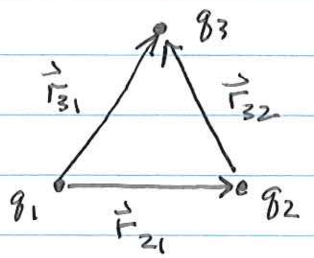
Note: negative direction of $\vec{ds}$ is taken into account by integrating r in negative direction ($dr < 0$)

When q_1 & q_2 have same sign, takes positive work to bring them together ($U > 0$)

When q_1 & q_2 have opposite sign, takes positive work to separate them since $U < 0$.

Potential energy from two point charges: $\boxed{U_{12} = \frac{q_1 q_2}{4\pi\varepsilon_0 r_{12}}}$

example: consider three charges q_1, q_2, q_3. What is U?



Just need to compute extra work to bring in q_3.

Use superposition principle: total force on q_3 is $\vec{F}_3 = \vec{F}_{21} + \vec{F}_{32}$ (sum of individual forces from q_1, q_2)

Work is $W = -\int \vec{F}_3 \cdot \vec{ds} = -\int \vec{F}_{31} \cdot \vec{ds} - \int \vec{F}_{32} \cdot \vec{ds}$

$$= U_{13} + U_{23} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{31}} + \frac{q_2 q_3}{r_{32}}\right)$$

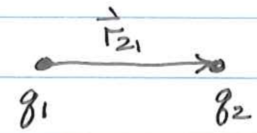
Total potential energy is: $U = U_{12} + U_{13} + U_{23}$

More general expression for N point charges:

$$\boxed{U = \sum_{i=1}^{N} \sum_{j=i+1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i q_j}{r_{ij}}}$$

Potential energy in electrostatics: total work required to "assemble" charge distribution.

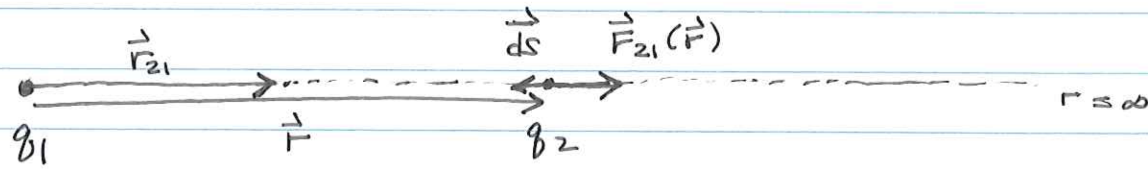
example: two charges separated by distance r_{21}. What is U?



step 0: all charges infinitely far away from one another $U = 0$.

Step 1: bring in q_1 from ∞. Takes no work since no forces acting on q_1. $U = 0$.

step 2: bring in q_2 from ∞. Requires work due to $\vec{F}_{21}$.



choose simplest path: q_2 moves in along radial direction
initial starting point: $|\vec{r}| = \infty$
end point: $\vec{r} = \vec{r}_{21}$

Force along path: $\vec{F}_{21}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r}$

Integration: $\int \vec{ds} = \int \hat{r}\, dr$